

# ZERO PATTERNS AND UNITARY SIMILARITY

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**ABSTRACT.** A subspace of the space,  $L(n)$ , of traceless complex  $n \times n$  matrices can be specified by requiring that the entries at some positions  $(i, j)$  be zero. The set,  $I$ , of these positions is a (zero) pattern and the corresponding subspace of  $L(n)$  is denoted by  $L_I(n)$ . A pattern  $I$  is universal if every matrix in  $L(n)$  is unitarily similar to some matrix in  $L_I(n)$ . The problem of describing the universal patterns is raised, solved in full for  $n \leq 3$ , and partial results obtained for  $n = 4$ . Two infinite families of universal patterns are constructed. They give two analogues of Schur's triangularization theorem.

## 1. INTRODUCTION

This paper is a sequel to our paper [1] where we studied the universal subspaces  $V$  for the representation of a connected compact Lie group  $G$  on a finite-dimensional real vector space  $U$ . The meaning of the word “universal” in this context is that every  $G$ -orbit in  $U$  meets the subspace  $V$ . The general results obtained in that paper have been applied in particular to the conjugation actions  $A \rightarrow XAX^{-1}$ ,  $X \in G$ , of the classical compact Lie groups  $G$ , i.e.,  $U(n)$ ,  $SO(n)$  and  $Sp(n)$ , on the space of  $n \times n$  matrices  $M(n, \mathbf{C})$ ,  $M(n, \mathbf{R})$  and  $M(n, \mathbf{H})$ , respectively. (By  $\mathbf{H}$  we denote the algebra of real quaternions.)

In the present paper we restrict our scope to the complex case, i.e., to  $M(n) = M(n, \mathbf{C})$  and  $G = U(n)$ . However, all results where we establish the nonsingularity (see Section 3 for the definition) of certain patterns are directly applicable to the real and quaternionic cases. Throughout the paper we denote by  $L(n) \subseteq M(n)$  the subspace of traceless matrices, and by  $T_n \subseteq U(n)$  the maximal torus consisting of the diagonal matrices. We shall consider only a very special class of complex subspaces of  $M(n)$ ; those that can be specified by requiring that the matrix entries in specified positions  $(i, j)$  vanish. We denote

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the set of these positions  $(i, j)$  by  $I$  and denote by  $M_I(n)$  the corresponding subspace. We also set  $L_I(n) = L(n) \cap M_I(n)$ . We refer to  $I$  as a (zero) pattern and denote the set of all such  $I$ 's by  $\mathcal{P}_n$ . A pattern is strict if it contains no diagonal positions. It is proper if it does not contain all the diagonal positions. We say that a pattern  $I \in \mathcal{P}_n$  is universal if the subspace  $L_I(n)$  is universal in  $L(n)$ . We point out that, for a strict pattern  $I \in \mathcal{P}_n$ ,  $L_I(n)$  is universal in  $L(n)$  iff  $M_I(n)$  is universal in  $M(n)$ .

The main question we consider, the universality problem, is to determine all universal patterns in  $\mathcal{P}_n$ . In full generality, this problem is solved only for  $n \leq 3$ . There is a simple necessary condition for universality of a proper pattern  $I$ :  $|I| \leq \mu_n = n(n-1)/2$  (see Proposition 2.4 below). We denote by  $\mathcal{P}'_n$  the set of strict patterns  $I \in \mathcal{P}_n$  with  $|I| = \mu_n$ . Theorem 5.1 of [1] provides a sufficient condition for the universality of a pattern (see Theorem 3.8 below). We use this result to construct some infinite families of strict universal patterns. The main results in this direction are Theorems 5.1 and 6.2.

In Section 2 we define the universal patterns and state the universality problem for patterns  $I \in \mathcal{P}_n$ . The case  $n = 2$  is easy: All patterns  $I \in \mathcal{P}_2$  of size 1 are universal. The nonsingularity of all  $I \in \mathcal{P}'_3$  has been established in [1]. In Proposition 2.6 we show that none of the proper nonstrict patterns  $I \in \mathcal{P}_3$  of size 3 is universal. Thus the universality problem is solved for  $n \leq 3$ . A proper pattern  $I \in \mathcal{P}_n$  is  $n$ -defective if the stabilizer of  $L_I(n)$  in  $U(n)$  has dimension larger than  $n^2 - 2|I|$ . Such patterns are not universal.

In Section 3 we introduce the nonsingular patterns. We say that  $I \in \mathcal{P}_n$  is  $n$ -nonsingular if  $\chi_I \notin \mathcal{K}(n)$ , where  $\chi_I \in \mathbf{R}[x_1, \dots, x_n]$  is the product of all differences  $x_i - x_j$ ,  $(i, j) \in I$ , and  $\mathcal{K}(n)$  is the ideal generated by the nontrivial elementary symmetric functions of the  $x_i$ 's. The basic fact, that nonsingular patterns are universal, was proved in [1]. We say that a pattern  $I$  is simple if  $(i, j) \in I$  implies that  $(j, i) \notin I$ . All simple patterns are nonsingular, and so universal. A pattern  $I \in \mathcal{P}'_n$  is  $n$ -exceptional if it is  $n$ -singular but not  $n$ -defective. There is no general method for deciding whether an exceptional pattern  $I \in \mathcal{P}'_n$  is universal. Proposition 3.3 provides a simple method for testing whether a pattern  $I \in \mathcal{P}'_n$  is nonsingular. The inner product  $\langle \cdot, \cdot \rangle$  used in the proposition is defined in the beginning of the section.

In Section 4 we introduce two equivalence relations “ $\approx$ ” and “ $\sim$ ” in  $\mathcal{P}'_n$ . We refer to the former simply as “equivalence” and to the latter as “weak equivalence”. This is justified since  $I \approx I'$  implies  $I \sim I'$ . If  $I \approx I'$  then  $I$  is universal iff  $I'$  is universal, but we do not know if this also holds for weak equivalence. However, if  $I \sim I'$  then  $I$  is nonsingular

iff  $I'$  is nonsingular. For any pattern  $I$  we define its complexity  $\nu(I)$  as the number of positions  $(i, j)$  with  $i \leq j$  such that both  $(i, j)$  and  $(j, i)$  belong to  $I$ . The patterns of complexity 0 are exactly the simple patterns. We show that for  $n \geq 4$  the set of patterns of complexity 1 in  $\mathcal{P}'_n$  splits into two weak equivalence classes. One of these classes is singular and the other nonsingular.

In Section 5 we consider a particular sequence of nonsingular patterns  $\Lambda_n \in \mathcal{P}'_n$ ,  $n \geq 1$ , of maximal complexity, i.e., with  $\nu(\Lambda_n) = \lfloor \mu_n/2 \rfloor$ . For convenience let us write  $n = 4m + r$  where  $m, r \geq 0$  are integers and  $r < 4$ . The pattern  $\Lambda_n$  consists of all positions  $(i, j)$  with  $i \neq j$  and  $i + j \leq n + 1$ , except those of the form  $(2i - 1, n - 2i + 1)$  and  $(n - 2i + 1, 2i - 1)$  with  $1 \leq i \leq m$ , and if  $r = 2$  or  $3$  we also omit the position  $(2m + 2, 2m + 1)$ . As  $\Lambda_n$  is nonsingular, it is also universal, i.e., every matrix  $A \in M(n)$  is unitarily similar to one in the subspace  $M_{\Lambda_n}(n)$ . Thus we can view this result as an analogue of Schur's theorem. The whole section is dedicated to the proof of this result.

In Section 6 we consider an infinite family of patterns  $J(\sigma, \mathbf{i})$  depending on an integer  $n \geq 1$ , a permutation  $\sigma \in S_n$  and a sequence  $\mathbf{i} = (i_1, i_2, \dots, i_{n-1})$  of distinct integers. This sequence has to be chosen so that, for each  $k$ ,  $|i_k| \in \{\sigma(1), \sigma(2), \dots, \sigma(k)\}$ . The pattern  $J(\sigma, \mathbf{i})$  consists of all positions  $(i_k, \sigma(j))$  with  $i_k > 0$  and  $(\sigma(j), -i_k)$  with  $i_k < 0$ , where in both cases  $1 \leq k < j \leq n$ . The main result of this section (Theorem 6.2) shows that all the patterns  $J(\sigma, \mathbf{i})$  are nonsingular. As a special case, we obtain another analogue of Schur's theorem (see Proposition 6.4).

In Section 7 we consider the exceptional patterns  $I \in \mathcal{P}'_4$ . Up to equivalence, there are seven of them (see Table 2). We prove that the first two of them are not universal while the third one is. This is the unique example that we have of a strict pattern which is singular and universal. For the remaining four patterns in Table 2 the universality question remains open. The same question for the nonstrict patterns in  $\mathcal{P}_4$  remains wide open.

There are other interesting questions that one can raise about the subspaces  $L_I(n)$  and the unitary orbits  $\mathcal{O}_A = \{UAU^{-1} : U \in \mathrm{U}(n)\}$ ,  $A \in L(n)$ . For instance, if  $L_I(n)$  is not universal we can ask for the characterization of the set  $\mathrm{U}(n) \cdot L_I(n)$ . Another question of interest is to determine the number,  $N_{A,I}$ , of  $T_n$ -orbits contained in the intersection  $\mathcal{X}_A = L_I(n) \cap \mathcal{O}_A$ . For instance, if  $L_I(n)$  is the space of upper (or lower) triangular traceless matrices and  $A \in L(n)$  has  $n$  distinct eigenvalues then  $N_{A,I} = n!$ .

In Section 8 we consider a pattern  $I \in \mathcal{P}'_n$  and a matrix  $A \in L(n)$  such that  $\mathcal{X}_A \neq \emptyset$ . The homogeneous space  $\mathcal{F}_n = \mathrm{U}(n)/T_n$  is known as the flag manifold and we refer to its points as flags. If  $g^{-1}Ag \in L_I(n)$ ,  $g \in \mathrm{U}(n)$ , we say that the flag  $gT_n$  reduces  $A$  to  $L_I(n)$ . We say that  $A$  is generic if  $\mathcal{X}_A$  and  $L_I(n)$  intersect transversally (see the next section for the definition). For generic  $A$ , we show that  $N_{A,I}$  is equal to the number of flags which reduce  $A$  to  $L_I(n)$ .

In Section 9 we consider the case  $n = 3$  and the cyclic pattern  $I = \{(1, 3), (2, 1), (3, 2)\}$ . We know that  $I$  is nonsingular and so  $\mathrm{U}(3) \cdot L_I(3) = L(3)$ . We show that the set  $\Theta$  of nongeneric matrices  $A \in L(3)$  is contained in a hypersurface  $\Gamma$  defined by  $P = 0$ , where  $P$  is a homogeneous  $\mathrm{U}(3)$ -invariant polynomial of degree 24. This polynomial is explicitly computed and we show that it is absolutely irreducible. The restriction,  $P_I$ , of  $P$  to  $L_I$  factorizes as  $P_I = P_1^2 P_2$ , where  $P_1$  and  $P_2$  are absolutely irreducible homogeneous polynomials of degree 6 and 12, respectively. Thus,  $\Gamma \cap L_I = \Gamma_1 \cup \Gamma_2$  where  $\Gamma_i \subset L_I$  is the hypersurface defined by  $P_i = 0$ ,  $i = 1, 2$ . The hypersurface  $\Gamma_1$  consists of all matrices  $A \in L(3)$  such that  $\mathcal{O}_A$  and  $L_I$  meet non-transversally at  $A$ .

We propose a conjecture (see Section 3) and several open problems.

For any positive integer  $n$  we set  $\mathbf{Z}_n = \{1, 2, \dots, n\}$  and  $\mu_n = n(n-1)/2$ . If  $\mathcal{A}$  is a  $\mathbf{Z}$ -graded algebra, we denote by  $\mathcal{A}_d$  the homogeneous component of  $\mathcal{A}$  of degree  $d$ . We use the same notation for the homogeneous ideals of  $\mathcal{A}$ .

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## 2. UNIVERSAL PATTERNS

Let  $M(n)$  denote the algebra of complex  $n \times n$  matrices and  $L(n)$  its subspace of matrices of trace 0. We are interested in subspaces of  $M(n)$  or  $L(n)$  which can be specified by zero patterns. For that purpose we introduce the notion of patterns.

A *position* is an ordered pair  $(i, j)$  of positive integers. We say that a position  $(i, j)$  is *diagonal* if  $i = j$ . A *pattern* is a finite set of positions. A pattern is *strict* if it has no diagonal positions. The *size* of a pattern  $I$  is its cardinality,  $|I|$ . If  $I, I'$  are patterns and  $I \subseteq I'$  then we say that  $I'$  is an *extension* of  $I$ . We denote by  $\mathcal{P}$  the set of all patterns and by  $\mathcal{P}_n$  the set of patterns contained in  $\mathbf{Z}_n \times \mathbf{Z}_n$ . We denote by  $\mathcal{P}'_n$  the subset of  $\mathcal{P}_n$  consisting of the strict patterns of size  $\mu_n$ . The “ $n \times n$  zero patterns” used in our paper [1] are the same as the patterns in  $\mathcal{P}'_n$ .

We define an involutory map  $T : \mathcal{P} \rightarrow \mathcal{P}$ , called *transposition*, by setting  $I^T = \{(j, i) : (i, j) \in I\}$  for  $I \in \mathcal{P}$ . We refer to  $I^T$  as the *transpose* of  $I$ . We say that  $I$  is *symmetric* if  $I^T = I$ . The sets  $\mathcal{P}_n$  and  $\mathcal{P}'_n$  are  $T$ -invariant for all  $n$ . Note that if  $I \in \mathcal{P}_n$  is universal (or nonsingular) then  $I^T$  has the same property.

For  $I \in \mathcal{P}_n$ , we denote by  $M_I(n)$ , or just  $M_I$  if  $n$  is fixed, the subspace of  $M(n)$  which consists of all matrices  $X = [x_{ij}]$  such that  $x_{ij} = 0$  for all  $(i, j) \in I$ . We also set  $L_I(n) = L(n) \cap M_I(n)$ .

Some important patterns in  $\mathcal{P}_n$  are the diagonal pattern  $\Delta_n = \{(i, i) : i \in \mathbf{Z}_n\}$  and the four triangular patterns:

$$\begin{aligned} \text{NE}_n &= \{(i, j) \in \mathbf{Z}_n \times \mathbf{Z}_n : i < j\}, \\ \text{SW}_n &= \{(i, j) \in \mathbf{Z}_n \times \mathbf{Z}_n : i > j\}, \\ \text{NW}_n &= \{(i, j) \in \mathbf{Z}_n \times \mathbf{Z}_n : i + j < n + 1\}, \\ \text{SE}_n &= \{(i, j) \in \mathbf{Z}_n \times \mathbf{Z}_n : i + j > n + 1\}. \end{aligned}$$

The first is the *upper triangular* and the second the *lower triangular* pattern. Note that, according to this terminology, if  $I$  is the upper triangular pattern, then  $M_I$  is the space of lower triangular matrices.

The unitary group  $U(n)$  acts on  $L(n)$  by conjugation, i.e., unitary similarities.

**Definition 2.1.** We say that a real subspace  $V \subseteq L(n)$  is *universal* in  $L(n)$  if every matrix in  $L(n)$  is unitarily similar to a matrix in  $V$ . We also say that a pattern  $I \in \mathcal{P}_n$  is *n-universal* if the subspace  $L_I(n)$  is universal in  $L(n)$ .

The prefix “ $n$ -” will be suppressed if  $n$  is clear from the context. This convention shall apply to several other definitions that we will introduce later.

It is obvious that a strict  $n$ -universal pattern is also  $m$ -universal for all  $m > n$ . The converse is not valid, e.g., the pattern  $\{(1, 2), (2, 1)\}$  is 3-universal but not 2-universal.

We are interested in the (pattern) universality problem, i.e., the problem of deciding which patterns  $I \in \mathcal{P}_n$  are universal. It is easy to see that, for a strict pattern  $I \in \mathcal{P}_n$ , the subspace  $L_I(n)$  is universal in  $L(n)$  iff the subspace  $M_I(n)$  is universal in  $M(n)$ . The Schur’s triangularization theorem asserts that the triangular patterns  $\text{NE}_n$  and  $\text{SW}_n$  are  $n$ -universal.

It is well known that  $\Delta_n$  is  $n$ -universal [5, Theorem 1.3.4] for all  $n$ . However, if  $i, j \in \mathbf{Z}_n$  and  $i \neq j$  then the next example implies that the pattern  $\Delta_n \cup \{(i, j)\}$  is not universal.

**Example 2.2.** The pattern  $I = \{(1, 1), (1, 2), (2, 2)\}$  is not  $n$ -universal for  $n \geq 2$ . Let  $D = \text{diag}(1, \dots, 1, 1 - n) \in L(n)$  and let  $A \in L_I(n)$ . As the rank of  $D - \text{id}$  is 1 and that of  $A - \text{id}$  is at least 2,  $D$  and  $A$  are not similar. Consequently,  $L_I$  is not universal. This implies that  $\text{NW}_n$  is not  $n$ -universal for  $n \geq 4$ . For the case  $n = 3$  see Proposition 2.6.

We give another example of a nonuniversal nonstrict pattern.

**Example 2.3.** The pattern  $J = \{(i, 1) : 1 \leq i < n\} \cup \{(1, n)\}$  is not  $n$ -universal for  $n \geq 2$ . Let  $D$  be as above and let  $A = [a_{ij}] \in L_I(n)$ . Assume that  $D$  and  $A$  are similar. Since  $D - \text{id}$  has rank 1,  $A - \text{id}$  must also have rank 1. Thus

$$\begin{vmatrix} -1 & 0 \\ a_{n1} & a_{nn} - 1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} -1 & a_{1j} \\ 0 & a_{jj} - 1 \end{vmatrix} = 0 \text{ for } 1 < j < n,$$

i.e.,  $a_{jj} = 1$  for  $1 < j \leq n$ . As  $A \in L_J$ , we have  $a_{11} = 0$  contradicting the fact that  $\text{tr}(A) = 0$ . Consequently,  $L_I$  is not universal.

If  $n - 1$  of the diagonal entries of a matrix  $A \in L(n)$  vanish then all  $n$  of them vanish. For that reason we introduce the following definition: We say that a pattern  $I \in \mathcal{P}_n$  is *n-proper* if  $(i, i) \notin I$  for at least one  $i \in \mathbf{Z}_n$ . Observe that, for any pattern  $I \in \mathcal{P}_n$ , the subspace  $L_I(n)$  is stabilized by the maximal torus  $T_n$ . An easy dimension argument shows that the following is valid, see [1, Lemma 4.1].

**Proposition 2.4.** *Let  $I \in \mathcal{P}_n$  be proper and universal. Then the dimension of the stabilizer of  $L_I(n)$  in  $\text{U}(n)$  does not exceed  $n^2 - 2|I|$ . In particular,  $n \leq n^2 - 2|I|$ , i.e.,  $|I| \leq \mu_n$ .*

Thus we have a simple condition that any proper universal pattern  $I \in \mathcal{P}_n$  must satisfy:  $|I| \leq \mu_n$ .

We say that a proper pattern  $I \in \mathcal{P}_n$  is *n-defective* if the dimension of the stabilizer of  $L_I(n)$  in  $\text{U}(n)$  is larger than  $n^2 - 2|I|$ . By the proposition, such patterns are not universal. Note that any proper pattern  $I \in \mathcal{P}_n$  with  $|I| > \mu_n$  is defective.

Next, we show that some special extensions of strict universal patterns are also universal. For  $I \in \mathcal{P}$  and integers  $m, n \geq 0$  we denote by  $(m, n) + I$  the translate  $\{(m + i, n + j) : (i, j) \in I\}$  of  $I$ .

**Proposition 2.5.** *Let  $I \in \mathcal{P}_n$  and  $J \in \mathcal{P}_{m-n}$ ,  $m > n$ , be strict patterns and assume that  $I$  is  $n$ -universal and  $J$  is  $(m - n)$ -universal. Then the pattern*

$$I' = I \cup ((n, n) + J) \cup ((0, n) + \mathbf{Z}_n \times \mathbf{Z}_{m-n})$$

*is  $m$ -universal.*

*Proof.* Given a matrix  $A \in L(m)$ , choose  $X \in U(m)$  such that  $B = XAX^{-1}$  is lower triangular. Let  $B_1$  resp.  $B_2$  denote the square submatrix of  $B$  of size  $n$  resp.  $m - n$  in the upper left resp. lower right hand corner. Since  $I$  and  $J$  are strict and universal, there exist matrices  $Y_1 \in U(n)$  and  $Y_2 \in U(m - n)$  such that  $Y_1 B_1 Y_1^{-1} \in M_I(n)$  and  $Y_2 B_2 Y_2^{-1} \in M_J(m - n)$ . Thus if  $Y = Y_1 \oplus Y_2$ , then  $YBY^{-1} \in L_{I'}(m)$ . Hence  $I'$  is  $m$ -universal.  $\square$

Let us fix a positive integer  $n$  and a pattern  $I \in \mathcal{P}'_n$ . For  $A \in L(n)$  we denote by  $\mathcal{O}_A$  the  $U(n)$ -orbit through  $A$ , i.e.,  $\mathcal{O}_A = \{UAU^{-1} : U \in U(n)\}$ . The set  $\mathcal{X}_A = L_I(n) \cap \mathcal{O}_A$  is closed and  $T_n$ -invariant. We say that  $\mathcal{O}_A$  intersects  $L_I$  *transversally* at a point  $B \in \mathcal{X}_A$  if the sum of  $L_I(n)$  and the tangent space of  $\mathcal{O}_A$  at  $B$  is equal to the whole space  $L(n)$ . If this is true for all points  $B \in \mathcal{X}_A$ , then we say that  $\mathcal{O}_A$  and  $L_I$  intersect *transversally*, and that the matrix  $A$  and its orbit  $\mathcal{O}_A$  are  *$I$ -generic*. We shall denote by  $N_{A,I}$  the cardinality of the set  $\mathcal{X}_A/T_n$  (the set of  $T_n$ -orbits in  $\mathcal{X}_A$ ). We note that  $N_{A,I}$  is finite if  $A$  is  $I$ -generic, see Section 8 and [1, Section 4].

Almost nothing is known about the universality of the subspaces  $L_I(n)$  of  $L(n)$  for nonstrict patterns  $I$ , but see the above examples. The case  $n = 2$  is easy and we leave it to the reader. Let us analyze the case  $n = 3$ . The case of strict patterns,  $\mathcal{P}'_3$ , has been handled in [1]. It is easy to see that any pattern  $I \in \mathcal{P}_3$  of size 2 is universal. By taking into account the above examples and the fact that  $\Delta_3$  is universal, there are only four cases to consider:

$$\begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & * \\ * & * & * \\ * & 0 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & * \\ * & * & 0 \\ * & * & * \end{bmatrix}.$$

(The starred entries are arbitrary, subject only to the condition that the matrices must have zero trace.)

We shall prove that none of them is universal and thereby complete the solution of the universality problem for  $n = 3$ .

**Proposition 2.6.** *For  $n = 3$ , no proper nonstrict pattern of size 3 is universal.*

*Proof.* We need only consider the four subspaces,  $V$ , mentioned above. It turns out that in all four cases there exists a diagonal matrix  $D \in L(3)$  such that  $\mathcal{O}_D \cap V = \emptyset$ .

In the first two cases the proof consists in constructing an  $U(3)$ -invariant polynomial function  $P : L(3) \rightarrow \mathbf{R}$  which is nonnegative on  $V$  and negative on the diagonal matrices  $D = \text{diag}(u, v, -u - v)$  when  $u$  and  $v$  are linearly independent over  $\mathbf{R}$ . The polynomial  $P$  will be

expressed as a polynomial in the  $U(3)$ -invariants  $i_k$  given in Appendix A. We shall write  $u = u_1 + iu_2$ ,  $u_1, u_2 \in \mathbf{R}$ , and similarly for other variables.

For the first subspace we define

$$\begin{aligned} P = & (2i_4 + 3i_5 - i_6)^2 + 4i_1(i_2 - i_3)(i_1i_3 - 6i_5) + 4i_1^2(i_1i_5 + i_8 - i_7) \\ & + 4i_1i_2(i_6 - i_4) + 4i_8(5i_3 - 4i_2) + 16i_3^2(2i_2 - i_3) \\ & + 4i_7(2i_2 - 3i_3) + 4i_2^2(i_2 - 5i_3) + 8(i_1i_{11} - i_{13}). \end{aligned}$$

A computation using MAPLE shows that for

$$A = \begin{bmatrix} 0 & 0 & x \\ 0 & z & y \\ u & v & -z \end{bmatrix} \in V$$

we have

$$P(A) = (|u|^2 - |x|^2)^2 (|x - \bar{u}|^2 z_1^2 - 4(u_1x_2 + u_2x_1)z_1z_2 + |x + \bar{u}|^2 z_2^2)^2.$$

On the other hand, we have  $P(D) = -64(u_1v_2 - u_2v_1)^2$ .

For the second subspace we define  $P = i_1^2 + 4(i_3 - i_2)$ . One can easily verify that for

$$A = \begin{bmatrix} 0 & x & y \\ u & z & 0 \\ v & 0 & -z \end{bmatrix} \in V$$

we have

$$P(A) = (|u|^2 + |v|^2 - |x|^2 - |y|^2)^2,$$

while  $P(D) = -4(u_1v_2 - u_2v_1)^2$ .

For the remaining subspaces we have more elementary arguments.

In the third case let  $D = \text{diag}(1, \zeta, \zeta^2) \in L(3)$ ,  $\zeta = (-1 + i\sqrt{3})/2$ . Since  $D$  is a normal matrix, its field of values,  $F(D)$ , is the equilateral triangle with vertices  $1, \zeta, \zeta^2$ . Assume that  $D$  is unitarily similar to a matrix  $A = [a_{ij}] \in L_I$ . As  $A \in L_I$ ,  $a_{22}$  is an eigenvalue of  $A$  and so  $a_{22} \in \{1, \zeta, \zeta^2\}$ . As  $\text{tr}(A) = 0$  and  $a_{11} = 0$ , we deduce that  $a_{33} = -a_{22} \notin F(D)$ . Since  $F(D) = F(A)$  and  $a_{33}$  is a diagonal entry of  $A$ , we have a contradiction.

In the last case let  $D = \text{diag}(1, i, -1 - i) \in L(3)$ . Assume that  $D$  is unitarily similar to a matrix

$$A = \begin{bmatrix} 0 & 0 & x \\ y & z & 0 \\ u & v & -z \end{bmatrix} \in V.$$



Since  $D$  is a normal matrix, so is  $A$ . From  $AA^* = A^*A$  we obtain the system of equations

$$\begin{aligned} x\bar{z} &= z\bar{u}, & z\bar{y} &= -v\bar{u}, & y\bar{u} &= -2z\bar{v}, \\ |x|^2 &= |y|^2 + |u|^2, & |y| &= |v|. \end{aligned}$$

Assume that  $z = 0$ . Then  $uy = 0$ . As  $A$  must be nonsingular, we have  $y \neq 0$  and  $u = 0$ . Thus  $A^3$  is a scalar matrix. Since  $D^3$  is not, we have a contradiction.

We conclude that  $z \neq 0$ . The equation  $x\bar{z} = z\bar{u}$  implies that  $|x| = |u|$ , which entails that  $y = v = 0$ . Hence  $z$  is an eigenvalue of  $A$ , and so  $z \in \{1, i, -1 - i\}$ . By switching the first two rows (and columns) of  $A$ , we obtain the direct sum  $[z] \oplus B$  where  $B = \begin{bmatrix} 0 & x \\ u & -z \end{bmatrix}$ . Hence  $0 \in F(B)$ . This is a contradiction since  $B$  is normal, and so  $F(B)$  is the line segment joining two of the eigenvalues of  $D$ .  $\square$

The following theorem provides an infinite collection of universal patterns. It is an easy consequence of a result of Košir and Sethuraman proven in [4].

**Theorem 2.7.** *The pattern*

$$((0, 1) + \text{NE}_{n-1}) \cup \{(i, 1) : 2 < i \leq n\} \cup \{(n, 2)\}, \quad n \geq 3,$$

*is  $n$ -universal.*

*Proof.* Denote this pattern by  $J$ . Let  $A \in M(n)$  be arbitrary. By [4, Theorem A.4], there exists  $S \in \text{GL}(n, \mathbf{C})$  such that  $SA^*S^{-1} \in M_{J'}(n)$  and  $SAS^{-1} \in M_{J''}(n)$ , where  $J' = (1, 0) + \text{SW}_{n-1}$  and  $J'' = \{(i, 1) : 2 < i \leq n\} \cup \{(n, 2)\}$ . By [4, Remark 1], we can assume that  $S$  is unitary. As  $J = (J')^T \cup J''$ , we deduce that  $SA^*S^{-1} \in M_J(n)$ . Hence  $J$  is  $n$ -universal.  $\square$

### 3. NONSINGULAR PATTERNS

Before defining the singular and nonsingular patterns we introduce some preliminary notions.

Let  $\mathbf{R}[x_1, x_2, \dots]$  resp.  $\mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$  be the polynomial ring resp. Laurent polynomial ring in countably many commuting independent variables  $x_1, x_2, \dots$  over  $\mathbf{R}$ . We introduce an inner product,  $\langle \cdot, \cdot \rangle$ , in  $\mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$  by declaring that the basis consisting of the Laurent monomials is orthonormal. We also introduce the involution  $f \rightarrow f^*$  and the shift endomorphism  $\tau$  of this Laurent polynomial ring where, by definition,

$$f^*(x_1, x_2, \dots) = f(x_1^{-1}, x_2^{-1}, \dots)$$

and  $\tau(x_i) = x_{i+1}$  for all  $i \geq 1$ . For any Laurent polynomial  $f$ , we denote by  $\text{CT}\{f\}$  the constant term of  $f$ . It is easy to see that for  $f, g \in \mathbf{R}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$  we have

$$\langle f, g \rangle = \text{CT}\{f^*g\} = \text{CT}\{fg^*\}.$$

We shall denote by  $\partial_i$  the partial derivative with respect to the variable  $x_i$ . For any  $f \in \mathbf{R}[x_1, x_2, \dots]$  we set

$$\partial_f = f(\partial_1, \partial_2, \dots).$$

To each pattern  $I$  we associate the polynomial

$$\chi_I = \prod_{(i,j) \in I} (x_i - x_j),$$

and the differential operator

$$\partial_I = \partial_{\chi_I} = \prod_{(i,j) \in I} (\partial_i - \partial_j).$$

Next we introduce special notation for certain symmetric polynomials in the first  $n$  variables  $x_1, x_2, \dots, x_n$ :

$$(3.1) \quad \sigma_{k,n} = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad k \in \{0, 1, \dots, n\},$$

for the elementary symmetric functions, and

$$(3.2) \quad h_{k,n} = \sum_{d_1 + \dots + d_n = k} x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}, \quad k \in \{0, 1, \dots, n\},$$

for the complete symmetric functions.

We denote by  $\mathcal{H}(n)$  the quotient of  $\mathbf{R}[x_1, x_2, \dots, x_n]$  modulo the ideal  $\mathcal{K}(n)$  generated by the  $\sigma_{k,n}$ ,  $k \in \mathbf{Z}_n$ , and define  $\varphi_n : \mathbf{R}[x_1, x_2, \dots, x_n] \rightarrow \mathcal{H}(n)$  to be the natural homomorphism.

**Definition 3.1.** We say that  $I \in \mathcal{P}_n$  is  $n$ -singular if  $\varphi_n(\chi_I) = 0$ , and otherwise that it is  $n$ -nonsingular. We say that  $I \in \mathcal{P}'_n$  is  $n$ -exceptional if it is  $n$ -singular but not  $n$ -defective.

Note that if a pattern  $I$  contains a diagonal position then  $\chi_I = 0$  (in particular,  $I$  is singular).

**Lemma 3.2.** *Every  $n$ -nonsingular pattern is also  $m$ -nonsingular for all  $m > n$ .*

*Proof.* Let  $I \in \mathcal{P}_n$  be  $m$ -singular for some  $m > n$ . Then there exist polynomials  $f_k \in \mathbf{R}[x_1, x_2, \dots, x_m]$  such that

$$\chi_I(x_1, x_2, \dots, x_n) = \sum_{k=1}^m f_k(x_1, x_2, \dots, x_m) \sigma_{k,m}(x_1, x_2, \dots, x_m).$$

By setting  $x_{n+1} = \cdots = x_m = 0$ , we see that  $I$  is  $n$ -singular.  $\square$

We remark that the converse of this lemma is not valid. (All seven patterns listed in Table 2 of Section 7 are counter-examples for  $n = 4$ .)

To simplify the notation, we set

$$\chi_n = \chi_{\text{NE}_n} = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Its expansion is given by the well known formula

$$(3.3) \quad \chi_n = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n x_{\sigma(i)}^{n-i}.$$

Consequently, we have  $\langle \chi_n, \chi_n \rangle = n!$ .

The following result (see [1, Proposition 2.3]) provides a practical method for deciding whether a pattern is nonsingular. In particular, it implies that  $\text{NE}_n$  is nonsingular.

**Proposition 3.3.** *A pattern  $I \in \mathcal{P}'_n$  is  $n$ -singular iff  $\langle \chi_I, \chi_n \rangle = 0$ .*

We now state two problems concerning the inner product in this proposition.

**Problem 3.4.** Is it true that  $\langle \chi_I, \chi_n \rangle \leq n!$  for all  $I \in \mathcal{P}_n$  and that equality holds iff  $\chi_I = \chi_n$ ?

**Problem 3.5.** Is it true that  $\langle \chi_I, \chi_I \rangle \geq n!$  for all  $I \in \mathcal{P}'_n$  and that equality holds iff  $\chi_I = \pm \chi_n$ ?

The computations carried out for  $n \leq 5$  show that, in these cases, the answer is affirmative for both problems.

In connection with Theorem 2.7 we propose

**Conjecture 3.6.** *Let  $J_{k,n} \in \mathcal{P}'_n$ ,  $k \in \mathbf{Z}_{n-1}$ , be the union of the translate  $(0, 1) + \text{NE}_{n-1}$ , the product  $\{n\} \times \mathbf{Z}_k$ , and  $\{(i, 1) : k < i < n\}$ . Then*

$$\langle \chi_{J_{k,n}}, \chi_n \rangle = (-1)^{n-1} \binom{n}{k} \binom{n-2}{k-1}.$$

The assertion in the case  $k = 1$  will be proved in Example 6.3. The universality of  $J_{2,n}$  was proved in Theorem 2.7. The conjecture has been verified for  $n \leq 10$  and all  $k \in \mathbf{Z}_{n-1}$ , except for  $(n, k) = (10, 5)$  in which case our program ran out of memory.

We say that a pattern  $I$  is *simply laced*, or just *simple*, if  $I \cap I^T = \emptyset$ , and otherwise we say that  $I$  is *doubly laced*. In particular, a simple pattern is strict. Any simple pattern  $I \in \mathcal{P}_n$  can be extended to a simple pattern  $I' \in \mathcal{P}'_n$ . As  $\chi_I$  divides  $\chi_{I'}$  and  $\chi_{I'} = \pm \chi_n$ , any simple pattern is nonsingular.

The following two results are extracted from Proposition 4.3 and Theorem 4.2 of our paper [1].

**Proposition 3.7.** *Every nonsingular pattern  $I \in \mathcal{P}_n$  can be extended to a nonsingular pattern  $I' \in \mathcal{P}'_n$ .*

**Theorem 3.8.** (Generalization of Schur's triangularization theorem)  
*Every nonsingular pattern in  $\mathcal{P}_n$  is  $n$ -universal. In particular, every simple pattern in  $\mathcal{P}_n$  is  $n$ -universal*

In connection with these results we pose an open problem.

**Problem 3.9.** Let  $I$  be a strict  $n$ -universal pattern. Does  $I$  extend to a strict  $n$ -universal pattern of size  $\mu_n$ ?

If we replace the word “strict” with “proper” then Proposition 2.6 shows that the answer is negative.

A special case of the general inner product identity that we prove in the next theorem will be used in the proof of the subsequent proposition.

**Theorem 3.10.** *Let  $G$  be a connected compact Lie group,  $T$  a maximal torus with Lie algebra  $\mathfrak{t}$  and  $W$  the Weyl group. Denote by  $\mu$  be the number of positive roots and by  $\chi$  their product. Let  $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$  be the algebra of polynomial functions  $\mathfrak{t} \rightarrow \mathbf{R}$  with the degree gradation, and let  $\langle \cdot, \cdot \rangle$  be a  $W$ -invariant inner product on  $\mathcal{A}$ . If  $h \in \mathcal{A}$  is  $W$ -invariant, then*

$$\|\chi\|^2 \langle hf, h\chi \rangle = \|h\chi\|^2 \langle f, \chi \rangle, \quad \forall f \in \mathcal{A}_\mu.$$

*Proof.* We may assume that  $h \neq 0$ . For the linear functional  $L : \mathcal{A}_\mu \rightarrow \mathbf{R}$  defined by  $L(f) = \langle hf, h\chi \rangle$ , we have  $\sigma \cdot L = \text{sgn}(\sigma)L$  for all  $\sigma \in W$ . There is a unique  $g \in \mathcal{A}_\mu$  such that  $L(f) = \langle f, g \rangle$  for all  $f \in \mathcal{A}_\mu$ . Moreover,  $\sigma \cdot g = \text{sgn}(\sigma)g$  for all  $\sigma \in W$ . As the sign representation of  $W$  occurs only once in  $\mathcal{A}_\mu$ , we must have  $g = c\chi$  for some  $c \in \mathbf{R}$ . The equality  $L(\chi) = \langle \chi, c\chi \rangle$  completes the proof.  $\square$

The next proposition, a nonsingular analogue of Proposition 2.5, shows that some extensions of nonsingular patterns are nonsingular. In a weaker form, it was originally conjectured by Jiu-Kang Yu.

**Proposition 3.11.** *For  $I \in \mathcal{P}'_n$ ,  $J \in \mathcal{P}'_{m-n}$ ,  $m > n$ , and*

$$I' = I \cup ((n, n) + J) \cup ((0, n) + \mathbf{Z}_n \times \mathbf{Z}_{m-n})$$

*we have*

$$\langle \chi_{I'}, \chi_m \rangle = \binom{m}{n} \langle \chi_I, \chi_n \rangle \langle \chi_J, \chi_{m-n} \rangle.$$

*Proof.* We apply the above theorem to the case where  $G = \mathrm{U}(n) \times \mathrm{U}(m-n)$ . Then the algebra  $\mathcal{A}$  can be identified with the polynomial algebra  $\mathbf{R}[x_1, \dots, x_m]$  so that  $\mathbf{R}[x_1, \dots, x_n]$  resp.  $\mathbf{R}[x_{n+1}, \dots, x_m]$  is the corresponding algebra for  $\mathrm{U}(n)$  resp.  $\mathrm{U}(m-n)$ . The polynomial  $\chi$  factorizes as  $\chi = \chi_n \cdot \tau^n \chi_{m-n}$ , where  $\tau$  is the shift operator. We take  $f = \chi_I \cdot \tau^n \chi_J$  and for  $h$  we take the polynomial

$$\prod_{i=1}^n \prod_{j=n+1}^m (x_i - x_j),$$

which is obviously invariant under  $W = S_n \times S_{m-n}$ . Since  $h\chi = \chi_m$  and  $hf = \chi_{I'}$ , the theorem gives the identity

$$\|\chi\|^2 \langle \chi_{I'}, \chi_m \rangle = \|\chi_m\|^2 \langle f, \chi \rangle.$$

Since  $\|\chi\|^2 = n!(m-n)!$ ,  $\|h\chi\|^2 = \|\chi_m\|^2 = m!$ , and

$$\langle \chi_I \cdot \tau^n \chi_J, \chi_n \cdot \tau^n \chi_{m-n} \rangle = \langle \chi_I, \chi_n \rangle \langle \chi_J, \chi_{m-n} \rangle,$$

the assertion follows.  $\square$

#### 4. EQUIVALENCE AND WEAK EQUIVALENCE

The symmetric group  $S_n$  acts on  $\mathcal{P}_n$  by  $\sigma(I) = \{(\sigma(i), \sigma(j)) : (i, j) \in I\}$  for  $\sigma \in S_n$ . For  $\sigma \in S_n$  and  $I \in \mathcal{P}_n$  we have  $\sigma \cdot \chi_I = \chi_{\sigma(I)}$ . As  $\mathcal{P}'_n$  is  $S_n$ -invariant, we obtain an action on  $\mathcal{P}'_n$ . We denote by  $\tilde{S}_n$  the group of transformations of  $\mathcal{P}'_n$  generated by the action of  $S_n$  and the restriction of transposition  $T$  to  $\mathcal{P}'_n$ . (Note that this restriction commutes with the action of  $S_n$ .) As the inner product is  $S_n$ -invariant, we have

$$\langle \chi_{\sigma(I)}, \chi_n \rangle = \langle \chi_I, \sigma \cdot \chi_n \rangle = \mathrm{sgn}(\sigma) \langle \chi_I, \chi_n \rangle, \quad I \in \mathcal{P}'_n, \quad \sigma \in S_n.$$

We say that the patterns  $I, I' \in \mathcal{P}'_n$  are *equivalent* if they belong to the same orbit of  $\tilde{S}_n$ . If so, we shall write  $I \approx I'$ . We denote by  $[I]$  the equivalence class of  $I \in \mathcal{P}'_n$ .

Assume that  $I \approx I'$ . It is easy to see that  $I'$  is universal iff  $I$  is universal. Since  $\ker \varphi_n$  is invariant under permutations of the variables  $x_1, x_2, \dots, x_n$ , we deduce that  $I'$  is nonsingular iff  $I$  is nonsingular. We say that the class  $[I]$  is *singular*, *nonsingular*, *universal*, *defective* or *exceptional* if  $I$  has the same property. These terms are clearly well defined.

Let  $I \in \mathcal{P}$  be any pattern and let us fix a position  $(i, j) \in I$  such that  $(j, i) \notin I$ . Denote by  $I'$  the pattern obtained from  $I$  by replacing the position  $(i, j)$  with  $(j, i)$ . We shall refer to the transformation  $I \rightarrow I'$  as a *flip*.

**Problem 4.1.** Let  $I \in \mathcal{P}'_n$  be  $n$ -universal and let  $I \rightarrow I'$  be a flip. Is it true that  $I'$  is  $n$ -universal?

We say that the patterns  $I, I' \in \mathcal{P}_n$  are *weakly equivalent* if  $I$  can be transformed to  $I'$  by using flips and the action of  $S_n$ . If so, we shall write  $I \sim I'$ . We denote by  $[I]_w$  the weak equivalence class of  $I \in \mathcal{P}_n$ . As the transposition map  $\mathcal{P}_n \rightarrow \mathcal{P}_n$  can be realized by a sequence of flips, we have  $[I] \subseteq [I]_w$  for all  $I \in \mathcal{P}_n$ . Note that the nonsingularity property is preserved by weak equivalence.

Let us define the *complexity*,  $\nu(I)$ , of a pattern  $I$  as the number of positions  $(i, j) \in I \cap I^T$  with  $i \leq j$ . The patterns of complexity 0 are precisely the simple patterns. Observe that a pattern  $I \in \mathcal{P}'_n$  has complexity 1 iff there exist a unique 2-element subset  $\{i, j\} \subseteq \mathbf{Z}_n$  such that  $(i, j)$  and  $(j, i)$  belong to  $I$ . Similarly,  $I \in \mathcal{P}'_n$  has complexity 1 iff there exist a unique 2-element subset  $\{k, l\} \subseteq \mathbf{Z}_n$  such that neither  $(k, l)$  nor  $(l, k)$  is in  $I$ .

It is natural to ask which patterns  $I \in \mathcal{P}'_n$  of complexity 1 are universal or nonsingular. We shall now give a complete answer to the latter question. At the same time we classify, up to weak equivalence, the patterns in  $\mathcal{P}'_n$  having complexity 1.

**Theorem 4.2.** Let  $I \in \mathcal{P}'_n$ ,  $\nu(I) = 1$ , and let  $\{i, j\}$  resp.  $\{k, l\}$  be the unique 2-element subset of  $\mathbf{Z}_n$  such that  $(i, j), (j, i) \in I$  resp.  $(k, l), (l, k) \notin I$ .

(a) If  $i, j, k, l$  are not distinct then

$$I \sim (\text{NE}_n \setminus \{(1, 2)\}) \cup \{(3, 1)\}, \quad n \geq 3,$$

and we have  $\langle \chi_I, \chi_n \rangle = \pm n!/2$ .

(b) If  $i, j, k, l$  are distinct then  $I$  is singular and

$$I \sim (\text{NE}_n \setminus \{(1, 2)\}) \cup \{(4, 3)\}, \quad n \geq 4.$$

*Proof.* (a) We have, say,  $i = k$ . Let  $J = \sigma(I)$ , where  $\sigma \in S_n$  is chosen so that  $\sigma(i) = 1$ ,  $\sigma(j) = 3$  and  $\sigma(l) = 2$ . For each  $(r, s) \in J$  with  $r > s$  and  $(r, s) \neq (3, 1)$  we apply a flip to replace  $(r, s)$  with  $(s, r)$ . We obtain the desired pattern. Proposition 3.11 gives the formula for the inner product.

(b) The equivalence assertion is proved in the same manner as in (a) and, since this new pattern is defective,  $I$  must be singular.  $\square$

We conclude this section by providing some numerical data about the equivalence classes in  $\mathcal{P}'_n$  for  $2 \leq n \leq 5$ . For each  $n$ , Table 1 gives the cardinality of  $\mathcal{P}'_n$ , the number of equivalence classes, and the number of nonsingular, defective, and exceptional classes in that order. The last column records the number of weak equivalence classes.

**Table 1: Equivalence classes**

$n$	$ \mathcal{P}'_n $	Equ.	Nons.	Def.	Exc.	Weak
2	2	1	1	0	0	1
3	20	3	3	0	0	2
4	928	30	19	4	7	12
5	184956	880	619	66	195	110

For any  $n$  we have  $|\mathcal{P}'_n| = \binom{2\mu_n}{\mu_n}$ , but a formula for the number of (weak) equivalence classes is not known.

**Problem 4.3.** Find a formula for the number of (weak) equivalence classes in  $\mathcal{P}'_n$ .

## 5. AN ANALOGUE OF SCHUR'S THEOREM

Recall the patterns  $\Lambda_n \in \mathcal{P}'_n$ ,  $n \geq 1$ , defined in the Introduction. The zero entries required by  $\Lambda_n$  for  $n = 2, 3, \dots, 7$  are exhibited below:

$$\begin{aligned}
 & \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & * & * \\ * & 0 & * & * & * \\ 0 & * & * & * & * \end{bmatrix}, \\
 & \begin{bmatrix} * & 0 & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & * & * & * & * \\ * & 0 & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ * & 0 & * & * & * & * & * \\ 0 & * & * & * & * & * & * \end{bmatrix}.
 \end{aligned}$$

The maximum of  $\nu(I)$  over all  $I \in \mathcal{P}'_n$  is  $\lfloor \mu_n/2 \rfloor$ . Let us write  $n = 4m + r$  where  $m$  is a nonnegative integer and  $r \in \{0, 1, 2, 3\}$ . Note that  $\Lambda_n$  is symmetric for  $r \in \{0, 1\}$ , and otherwise there is a unique  $(i, j) \in \Lambda_n$ , namely  $(2m + 1, 2m + 2)$ , such that  $(j, i) \notin \Lambda_n$ . Hence  $\Lambda_n$  has the maximal complexity  $\lfloor \mu_n/2 \rfloor$ . Our objective in this section is to prove that  $\Lambda_n$  is nonsingular. As  $\Lambda_n$  is a (necessary) minor modification of the triangular pattern  $\text{NW}_n$ , we consider this result as an analogue of Schur's theorem.

**Theorem 5.1.** *We have  $\langle \chi_{\Lambda_n}, \chi_n \rangle = (-1)^s n! / 2^s$ , where  $s = \lfloor (n+1)/4 \rfloor$ . In particular,  $\Lambda_n$  is universal.*

For the proof we need four lemmas and the following three facts which follow immediately from [3, Chapter III, Lemma 3.9]:

- (i)  $x_i^k \in \mathcal{K}(n)$  for  $k \geq n$ ,  $1 \leq i \leq n$ ;
- (ii) If  $f, g \in \mathbf{R}[x_1, x_2, \dots, x_n]$  and  $f \equiv g \pmod{\mathcal{K}(n)}$  then

$$(5.1) \quad \partial_f \chi_n = \partial_g \chi_n;$$

- (iii) If  $f \in \mathbf{R}[x_1, x_2, \dots, x_n]_{\mu_n}$  then

$$(5.2) \quad \partial_f \chi_n = \prod_{k=1}^{n-1} k! \cdot \langle f, \chi_n \rangle.$$

By differentiating the formula (3.2), we obtain that

$$(5.3) \quad \partial_i h_{k,n} = \sum_{d_1 + \dots + d_n = k-1} (1 + d_i) x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}, \quad i \leq n.$$

We prove first the following congruence.

**Lemma 5.2.** *For  $1 \leq r \leq m \leq n$  we have*

$$\prod_{m < i \leq n} (x_r - x_i) \equiv \partial_r h_{n-m+1, m} \pmod{\mathcal{K}(n)}.$$

*Proof.* Without any loss of generality we may assume that  $r = 1$ . Let  $s, t$  be two additional commuting indeterminates. Observe that for any polynomial  $f(t)$ , with coefficients in  $\mathbf{R}[x_1, \dots, x_n]$ , the constant term of

$$F(s, t) = f(s^{-1}) \cdot \frac{1}{1 - st},$$

when expanded into a formal Laurent series with respect to  $s$ , is equal to  $f(t)$ . For

$$f(t) = (t - x_1) \cdot \prod_{m < i \leq n} (t - x_i)$$

we have

$$\begin{aligned} F(s, t) &= \frac{\prod_{i=1}^n (1 - sx_i)}{s^{n-m+1} (1 - st) \prod_{i=2}^m (1 - sx_i)} \\ &= \frac{1}{s^{n-m+1}} \cdot \sum_{k=0}^n (-s)^k \sigma_{k,n} \cdot \sum_{l=0}^{\infty} h_{l,m}(t, x_2, \dots, x_m) s^l. \end{aligned}$$

Hence

$$f(t) = \sum_{k=0}^{n-m+1} (-1)^k \sigma_{k,n} h_{n-m+1-k, m}(t, x_2, \dots, x_m).$$



By evaluating the partial derivative with respect to  $t$  at the point  $t = x_1$ , we obtain that

$$\prod_{m < i \leq n} (x_1 - x_i) = \sum_{k=0}^{n-m+1} (-1)^k \sigma_{k,n} \partial_1 h_{n-m+1-k,m}(x_1, x_2, \dots, x_m)$$

and the assertion of the lemma follows.  $\square$

Recall the endomorphism  $\tau$  defined in the beginning of Section 3.

**Lemma 5.3.** *For  $P = \partial_1 h_{2,n-1}(x_1, x_n)$ , i.e.,*

$$P = \sum_{k=0}^{n-2} (n-1-k) x_1^{n-2-k} x_n^k, \quad n \geq 4,$$

we have

$$(5.4) \quad \partial_P^2 \chi_n = (-1)^n n! (n-2)! \left( \frac{n-1}{2} x_1 + \frac{n-3}{2} x_n - \sum_{k=2}^{n-1} x_k \right) \tau \chi_{n-2}.$$

*Proof.* Since  $x_1^n, x_n^n \in \mathcal{K}(n)$ , we have

$$P^2 \equiv a x_1^{n-3} x_n^{n-1} + b x_1^{n-2} x_n^{n-2} + c x_1^{n-1} x_n^{n-3}, \quad (\text{mod } \mathcal{K}(n)),$$

where

$$\begin{aligned} a &= \sum_{k=1}^{n-2} k(n-1-k) = n(n-1)(n-2)/6, \\ b &= \sum_{k=1}^{n-1} k(n-k) = n(n-1)(n+1)/6, \\ c &= \sum_{k=2}^{n-1} k(n+1-k) = n(n+5)(n-2)/6. \end{aligned}$$

By using the property (5.1), we obtain that

$$\partial_P^2 \chi_n = (a \partial_1^{n-3} \partial_n^{n-1} + b \partial_1^{n-2} \partial_n^{n-2} + c \partial_1^{n-1} \partial_n^{n-3}) \chi_n.$$

If we omit from  $\chi_n$  the terms  $\pm x_1^{d_1} \cdots x_n^{d_n}$  with  $d_1 + d_n < 2n-4$ , we obtain the polynomial  $(-1)^n Q \tau \chi_{n-2}$  where

$$Q = (x_1^{n-1} x_n^{n-2} - x_1^{n-2} x_n^{n-1}) - (x_1^{n-1} x_n^{n-3} - x_1^{n-3} x_n^{n-1}) \sum_{k=2}^{n-1} x_k.$$

Since the omitted terms are killed by  $\partial_P^2$ , we have  $\partial_P^2 \chi_n = (-1)^n C \tau \chi_{n-2}$  where

$$C = (n-1)! \left[ (n-2)!((b-a)x_1 + (c-b)x_n) - (n-3)!(c-a) \sum_{k=2}^{n-1} x_k \right].$$

It remains to plug in the values of  $a, b$  and  $c$ .  $\square$

The next lemma gives another important identity.

**Lemma 5.4.** *Let  $J_n = J'_n \cup (J'_n)^T$ ,  $n \geq 4$ , where  $J'_n$  is the union of  $\{(1, 2), (1, n-1), (2, n)\}$  and  $\{1, 2\} \times \{3, 4, \dots, n-2\}$ . Then*

$$\partial_{J_n} \chi_n = \frac{1}{2} n! (n-1)! (n-2)! (n-3)! \tau^2 \chi_{n-4}.$$

*Proof.* By using Lemma 5.2 we obtain the congruence

$$\begin{aligned} \chi_{J_n} &= - \left( \prod_{i=2}^{n-1} (x_1 - x_i) \right)^2 \left( (x_2 - x_n) \prod_{i=3}^{n-2} (x_2 - x_i) \right)^2 \\ &\equiv -P^2 R^2 \pmod{\mathcal{K}(n)}, \end{aligned}$$

where  $P$  is defined as in the previous lemma and

$$R = \partial_2 h_{n-2,3}(x_1, x_2, x_{n-1}) = \sum_{d_1+d_2+d_3=n-3} (1+d_2) x_1^{d_1} x_2^{d_2} x_{n-1}^{d_3}.$$

In view of the formula (5.4), it suffices to prove that

$$(5.5) \quad \partial_R^2 F = \frac{(-1)^{n-1}}{2} (n-1)! (n-3)! \tau^2 \chi_{n-4},$$

where

$$F = \left( \frac{n-1}{2} x_1 + \frac{n-3}{2} x_n - \sum_{k=2}^{n-1} x_k \right) \tau \chi_{n-2}.$$

Since  $\deg(R^2) = 2n-6$ , we need only consider the terms in  $F$  for which the sum of the exponents of  $x_1, x_2$  and  $x_{n-1}$  is at least  $2n-6$ . Their sum is

$$F' = (-1)^n \left( \frac{n-1}{2} x_1 - x_2 - x_{n-1} \right) (x_2^{n-3} x_{n-1}^{n-4} - x_2^{n-4} x_{n-1}^{n-3}) \tau^2 \chi_{n-4}.$$

Note also that the exponents of  $x_1$  in  $F'$  are  $\leq 1$ . So we need only consider the terms in  $\partial_R^2$  for which the exponent of  $\partial_1$  is 0 or 1. Their sum is  $D_1^2 + 2\partial_1 D_1 D_2$  where

$$D_1 = \sum_{k=0}^{n-3} (n-2-k) \partial_2^{n-3-k} \partial_{n-1}^k,$$

$$D_2 = \sum_{k=0}^{n-4} (n-3-k) \partial_2^{n-4-k} \partial_{n-1}^k.$$

After expanding the products  $D_1 D_2$  and  $D_1^2$ :

$$\begin{aligned} D_1 D_2 &= a_1 \partial_2^{n-3} \partial_{n-1}^{n-4} + b_1 \partial_2^{n-4} \partial_{n-1}^{n-3} + \dots, \\ D_1^2 &= a_2 \partial_2^{n-2} \partial_{n-1}^{n-4} + b_2 \partial_2^{n-4} \partial_{n-1}^{n-2} + \dots, \end{aligned}$$

the exhibited coefficients can be easily computed:

$$\begin{aligned} a_1 &= \sum_{k=1}^{n-3} k(n-1-k) = (n-2)(n-3)(n+2)/6, \\ a_2 &= \sum_{k=2}^{n-2} k(n-k) = (n-1)(n-3)(n+4)/6, \\ b_1 = b_2 &= \sum_{k=1}^{n-3} k(n-2-k) = (n-1)(n-2)(n-3)/6. \end{aligned}$$

Thus

$$\begin{aligned} D_1 D_2 (x_2^{n-3} x_{n-1}^{n-4} - x_2^{n-4} x_{n-1}^{n-3}) &= (n-3)!(n-4)!(a_1 - b_1) \\ &= (n-2)!(n-3)!/2, \\ D_1^2 (x_2^{n-2} x_{n-1}^{n-4} - x_2^{n-4} x_{n-1}^{n-2}) &= (n-2)!(n-4)!(a_2 - b_2) \\ &= (n-1)!(n-3)!. \end{aligned}$$

Hence

$$\begin{aligned} \partial_R^2 F &= (D_1^2 + 2\partial_1 D_1 D_2) F' \\ &= (-1)^n \left[ (n-1) D_1 D_2 (x_2^{n-3} x_{n-1}^{n-4} - x_2^{n-4} x_{n-1}^{n-3}) \right. \\ &\quad \left. - D_1^2 (x_2^{n-2} x_{n-1}^{n-4} - x_2^{n-4} x_{n-1}^{n-2}) \right] \tau^2 \chi_{n-4}, \end{aligned}$$

which completes the proof.  $\square$

The fourth lemma follows easily from the previous one.

**Lemma 5.5.** *Let  $I = J_n \cup ((2, 2) + I')$ ,  $n \geq 4$ , where  $J_n$  is defined as in the previous lemma and  $I' \in \mathcal{P}'_{n-4}$  is arbitrary. Then*

$$\langle \chi_I, \chi_n \rangle = \frac{1}{2} n(n-1)(n-2)(n-3) \langle \chi_{I'}, \chi_{n-4} \rangle.$$

*Proof.* Since  $\chi_I = \chi_{J_n} \tau^2 \chi_{I'}$ , we have  $\partial_I = \partial_{J_n} \partial_{\tau^2 \chi_{I'}}$ . By using (5.2) and Lemma 5.4, we obtain that

$$\begin{aligned} \langle \chi_I, \chi_n \rangle &= \frac{\partial_I \chi_n}{\prod_{k=1}^{n-1} k!} \\ &= \frac{n(n-1)(n-2)(n-3)\tau^2(\partial_{I'} \chi_{n-4})}{2 \prod_{k=1}^{n-5} k!} \\ &= \frac{1}{2} n(n-1)(n-2)(n-3) \langle \chi_{I'}, \chi_{n-4} \rangle. \end{aligned}$$

□

We can now prove the theorem itself.

*Proof.* We construct the patterns  $\Lambda'_n$  inductively as  $\Lambda'_0 = \Lambda'_1 = \emptyset$ ,  $\Lambda'_2 = \{(1, 2)\}$ ,  $\Lambda'_3 = \{(2, 1), (2, 3), (3, 2)\}$ , and  $\Lambda'_n = J_n \cup ((2, 2) + \Lambda'_{n-4})$  for  $n \geq 4$ . We claim that  $\langle \chi_{\Lambda'_n}, \chi_n \rangle = n!/2^s$ .

We prove the claim by induction on  $n$ . It is straightforward to verify the claim for  $n = 0, 1, 2, 3$ . For  $n \geq 4$ , by Lemma 5.5 and the induction hypothesis, we have

$$\begin{aligned} \langle \chi_{\Lambda'_n}, \chi_n \rangle &= \frac{1}{2} n(n-1)(n-2)(n-3) \langle \chi_{\Lambda'_{n-4}}, \chi_{n-4} \rangle \\ &= \frac{1}{2} n(n-1)(n-2)(n-3) \cdot \frac{(n-4)!}{2^{s-1}} = \frac{n!}{2^s}. \end{aligned}$$

It remains to observe that  $\sigma(\Lambda'_n) = \Lambda_n$ , where

$$\sigma = \prod_{k=1}^s (2k-1, 2k) \in S_n.$$

□

## 6. A REMARKABLE FAMILY OF NONSINGULAR PATTERNS

It is a challenging problem to construct an infinite family of new nonsingular doubly laced patterns. The main result of this section gives a construction of such a family. It includes, as a special case, a new analogue of Schur's theorem namely another modification of the triangular pattern  $NW_n$  (see Proposition 6.4).

We first introduce the notation that we need to state and prove our theorem. Let  $\sigma \in S_n$  and let  $\mathbf{r} = (r_1, r_2, \dots, r_{n-1})$  be a sequence with  $r_k \in \sigma(\mathbf{Z}_k)$  for all  $k \in \mathbf{Z}_{n-1}$ . For  $k \in \mathbf{Z}_{n-1}$  we set

$$I_k = \{(r_k, \sigma(j)) : k < j \leq n\}.$$

As usual,  $\delta_{ij}$  will denote the Kronecker delta symbol.

The main technical tool is the following lemma.

**Lemma 6.1.** *Under the above hypotheses, for  $0 \leq m < n$  we have*

$$\begin{aligned} \partial_{I_1} \partial_{I_2} \cdots \partial_{I_m} \chi_n = \\ \operatorname{sgn}(\sigma) \left( \prod_{k=1}^m (n - k + \delta_{r_k, \sigma(k)})! \right) \chi_{n-m}(x_{\sigma(m+1)}, \dots, x_{\sigma(n)}). \end{aligned}$$

*Proof.* We use induction on  $m$ . The assertion obviously holds for  $m = 0$ . Assume that  $m > 0$ . By the induction hypothesis we have

$$\begin{aligned} \partial_{I_1} \partial_{I_2} \cdots \partial_{I_m} \chi_n = \\ \operatorname{sgn}(\sigma) \left( \prod_{k=1}^{m-1} (n - k + \delta_{r_k, \sigma(k)})! \right) \cdot \partial_{I_m} \chi_{n-m+1}(x_{\sigma(m)}, \dots, x_{\sigma(n)}). \end{aligned}$$

By Lemma 5.2 we have

$$\begin{aligned} \chi_{I_m} &= \prod_{m < i \leq n} (x_{r_m} - x_{\sigma(i)}) \\ &\equiv \partial_{r_m} h_{n-m+1, m}(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \pmod{\mathcal{K}(n)}. \end{aligned}$$

By invoking the property (5.1) and the identity (5.2), we deduce that

$$\begin{aligned} \partial_{I_m} \chi_{n-m+1}(x_{\sigma(m)}, \dots, x_{\sigma(n)}) = \\ \sum_{d_1 + \cdots + d_m = n-m} (1 + d_{r_m}) \partial_{\sigma(1)}^{d_1} \cdots \partial_{\sigma(m)}^{d_m} \chi_{n-m+1}(x_{\sigma(m)}, \dots, x_{\sigma(n)}). \end{aligned}$$

All terms in this sum vanish except the one for  $d_1 = \cdots = d_{m-1} = 0$  and  $d_m = n - m$ . We infer that

$$\begin{aligned} \partial_{I_m} \chi_{n-m+1}(x_{\sigma(m)}, \dots, x_{\sigma(n)}) = \\ (1 + (n - m) \delta_{r_m, \sigma(m)}) \partial_{\sigma(m)}^{n-m} \chi_{n-m+1}(x_{\sigma(m)}, \dots, x_{\sigma(n)}). \end{aligned}$$

Since

$$\partial_{\sigma(m)}^{n-m} \chi_{n-m+1}(x_{\sigma(m)}, \dots, x_{\sigma(n)}) = (n - m)! \chi_{n-m}(x_{\sigma(m+1)}, \dots, x_{\sigma(n)}),$$

we are done.  $\square$

Let us fix a permutation  $\sigma \in S_n$  and let  $\mathbf{i} = (i_1, i_2, \dots, i_{n-1})$  be a sequence of distinct integers such that  $|i_k| \in \sigma(\mathbf{Z}_k)$  for all  $k \in \mathbf{Z}_{n-1}$ . Next we set

$$(i_k, \sigma(j))^+ = \begin{cases} (i_k, \sigma(j)) & \text{if } i_k > 0; \\ (\sigma(j), -i_k) & \text{otherwise.} \end{cases}$$

With these data at hand, we construct the strict pattern

$$(6.1) \quad J(\sigma, \mathbf{i}) = \{(i_k, \sigma(j))^+ : 1 \leq k < j \leq n\}.$$

We claim that the conditions imposed on  $\mathbf{i}$  imply that the map sending  $(k, j) \rightarrow (i_k, \sigma(j))^+$  for  $k < j$  is injective, i.e., that  $|J(\sigma, \mathbf{i})| = \mu_n$ . Indeed, assume that two different pairs  $(k, j)$  and  $(r, s)$ , with  $k < j$  and  $r < s$ , have the same image, i.e.,  $(i_k, \sigma(j))^+ = (i_r, \sigma(s))^+$ . Clearly, we must have  $k \neq r$  and  $i_k i_r < 0$ . Say,  $k < r$ . Then  $|i_k| = \sigma(s) \notin \sigma(\mathbf{Z}_r)$ , which contradicts the condition  $|i_k| \in \sigma(\mathbf{Z}_k)$ . This proves our claim, and so we have  $J(\sigma, \mathbf{i}) \in \mathcal{P}'_n$ .

Let  $\mathcal{J}_n \subseteq \mathcal{P}'_n$  be the set of all patterns  $J(\sigma, \mathbf{i})$ . For any  $n$ , let  $\iota \in S_n$  be the identity permutation. There are exactly  $(n!)^2$  choices for the ordered pairs  $(\sigma, \mathbf{i})$ . However the corresponding patterns  $J(\sigma, \mathbf{i})$  are not all distinct. For instance, if  $n = 2$  we have  $J(\sigma, (2)) = J(\iota, (-1))$  with  $\sigma = (1, 2)$ .

The following is the main result of this section.

**Theorem 6.2.** *For  $J = J(\sigma, \mathbf{i})$  we have*

$$\langle \chi_J, \chi_n \rangle = (-1)^d \operatorname{sgn}(\sigma) \prod_{k: |i_k| = \sigma(k)} (n - k + 1),$$

where

$$d = \sum_{k: i_k < 0} (n - k).$$

*Proof.* We apply the lemma with  $\mathbf{r} = (|i_1|, |i_2|, \dots, |i_{n-1}|)$  and  $m = n - 1$ . Thus we now have  $I_k = \{(|i_k|, \sigma(j)) : k < j \leq n\}$  for all  $k \in \mathbf{Z}_{n-1}$ . Since  $\chi_1 = 1$ , the lemma gives

$$\partial_{I_1} \partial_{I_2} \cdots \partial_{I_{n-1}} \chi_n = \operatorname{sgn}(\sigma) \prod_{k=1}^{n-1} (n - k + \delta_{|i_k|, \sigma(k)})!.$$

By (5.2) we have

$$\partial_{I_1} \partial_{I_2} \cdots \partial_{I_{n-1}} \chi_n = \prod_{k=1}^{n-1} (n - k)! \cdot \langle \chi_{I_1} \chi_{I_2} \cdots \chi_{I_{n-1}}, \chi_n \rangle.$$

Observe that  $J$  is the disjoint union of the  $I_k$ 's with  $i_k > 0$  and the  $I_k^T$ 's with  $i_k < 0$ . Therefore we have  $\chi_J = (-1)^d \chi_{I_1} \chi_{I_2} \cdots \chi_{I_{n-1}}$  and the assertion follows.  $\square$

Let us give an example.

**Example 6.3.** Let  $n > 1$  and let  $\sigma \in S_n$  be the identity. We set  $i_1 = -1$  and  $i_k = k - 1$  for  $1 < k < n$ . The  $i_k$ 's are distinct and the condition  $|i_k| \in \mathbf{Z}_k$  is satisfied for all  $k$ . In this case we obtain the pattern

$$J = \{(i, j) : 1 \leq i < j - 1 \leq n - 1\} \cup \{(i, 1) : 1 < i \leq n\}.$$

Only  $i_1$  is negative and so  $d = n - 1$ , and  $|i_k| = \sigma(k) = k$  is valid only for  $k = 1$ . By the theorem we have  $\langle \chi_J, \chi_n \rangle = (-1)^{n-1}n$ . This proves the case  $k = 1$  of Conjecture 3.6.

Recall that  $NW_n$  is not  $n$ -universal for  $n \geq 3$  (see Example 2.2). However, if we modify this pattern to make it strict by replacing its diagonal positions  $(i, i)$  with  $(i, n + 1 - i)$ , we can show that the new pattern

$$\Pi_n = \{(i, j) : i + j \leq n, i \neq j\} \cup \{(i, n - i + 1) : 2i \leq n\}$$

is nonsingular and, consequently, universal. This is our second analogue of Schur's theorem. The zero entries required by  $\Pi_n$  for  $n = 2, 3, 4, 5$  are exhibited below:

$$\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & * & * & * \\ * & * & * & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & * \\ 0 & 0 & * & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{bmatrix}.$$

**Proposition 6.4.** *We have  $\langle \chi_{\Pi_n}, \chi_n \rangle = n!!$ .*

*Proof.* This is in fact a special case of Theorem 6.2. We take  $\sigma \in S_n$  to be the permutation  $1, n, 2, n-1, \dots$ . Thus  $\sigma(2k-1) = k$  for  $2k-1 \leq n$  and  $\sigma(2k) = n+1-k$  for  $2k \leq n$ . We set  $\mathbf{i} = (1, -1, 2, -2, \dots)$ . The  $i_k$ 's are distinct. As  $i_{2k-1} = -i_{2k} = k = \sigma(2k-1)$ , the condition  $|i_k| \in \sigma(\mathbf{Z}_k)$  is satisfied for all  $k$ 's. With this  $\sigma$  and  $\mathbf{i}$  we have  $\Pi_n = J(\sigma, \mathbf{i})$ . The equality  $|i_k| = \sigma(k)$  holds iff  $k$  is odd and the inequality  $i_k < 0$  holds iff  $k$  is even. Thus  $d = \sum (n - 2k)$ , the sum being over all positive integers  $k$  such that  $2k \leq n$ . Therefore  $d$  is even for  $n$  even and  $d \equiv [n/2] \pmod{2}$  for  $n$  odd. One can easily verify that  $\text{sgn}(\sigma) = (-1)^d$  in all cases. Hence, we obtain the formula given in the proposition.  $\square$

In several cases we used [7] to identify various sequences that we have encountered, such as the double factorial sequence A006882 in the above proposition.

Let  $J = J(\sigma, \mathbf{i})$  be the pattern (6.1). Note that  $J^T = J(\sigma, -\mathbf{i})$ , where  $-\mathbf{i} = (-i_1, -i_2, \dots, -i_{n-1})$ . If  $\rho \in S_n$  it is easy to verify that  $\rho(J) = J(\rho\sigma, \mathbf{j})$  where  $\mathbf{j} = (j_1, j_2, \dots, j_{n-1})$  with

$$j_k = \begin{cases} \rho(i_k) & \text{if } i_k > 0; \\ -\rho(-i_k) & \text{otherwise.} \end{cases}$$

It follows that  $\mathcal{J}_n$  is a union of equivalence classes.

**Problem 6.5.** Determine the number of equivalence classes contained in  $\mathcal{J}_n$ .

For  $n = 2, 3, 4, 5$  the answers are 1, 2, 7, 34 respectively.

In the case when  $\sigma = \iota$ , the sequence  $\mathbf{i} = (i_1, i_2, \dots, i_{n-1})$  is subject only to the conditions: (a) the integers  $i_k$  are pairwise distinct and (b)  $i_k \in \mathbf{Z}_k$  for all  $k \in \mathbf{Z}_{n-1}$ . In particular  $i_1 = \pm 1$ . To simplify the notation, in this case we set  $J(\mathbf{i}) = J(\iota, \mathbf{i})$ . From the previous discussion it is clear that each equivalence class contained in  $\mathcal{J}_n$  has a representative of the form  $J(\mathbf{i})$  with  $i_1 = 1$ . However,  $J(\mathbf{i}) \approx J(\mathbf{j})$  may hold for two different sequences  $\mathbf{i}$  and  $\mathbf{j}$  with  $i_1 = j_1 = 1$ . For instance, for  $n = 3$  we have  $J((1, 2)) \approx J((1, -2))$ .

## 7. THE CASE $n = 4$

In this section we fix  $n = 4$ . Recall that the nonsingular patterns are universal, and the defective ones are not. In this section we shall exhibit a strict pattern which is singular and universal (see Proposition 7.3 below). There are 7 exceptional equivalence classes in  $\mathcal{P}'_4$ , their representatives are listed in Table 2. The last column of the table shows what is known about the universality of the pattern.

**Table 2: Exceptional classes for  $n = 4$**

No.	Representative pattern	Univ.
1	$\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$	No
2	$\{(1,2),(2,1),(1,3),(1,4),(2,4),(3,2)\}$	No
3	$\{(1,2),(1,3),(1,4),(2,1),(3,4),(4,3)\}$	Yes
4	$\{(1,2),(2,1),(1,3),(3,1),(2,4),(4,3)\}$	?
5	$\{(1,2),(2,1),(1,3),(2,3),(4,1),(4,2)\}$	?
6	$\{(1,2),(2,1),(1,4),(2,3),(3,1),(4,2)\}$	?
7	$\{(1,2),(2,1),(1,4),(3,1),(3,4),(4,3)\}$	?

We shall prove now that the first two patterns are not universal.

**Proposition 7.1.** *The first pattern in Table 2 is not universal.*

*Proof.* Denote this pattern by  $I$ . Let  $A \in L(4)$  be the matrix whose entries in positions  $(1, 2)$  and  $(3, 4)$  are 1 and all other entries are 0. Note that  $A$  has rank 2 and that  $A^2 = 0$ . Assume that  $A$  is unitarily



similar to some  $X = [x_{ij}] \in L_I$ . Thus  $X$  has the form

$$X = \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ * & * & * & * \end{bmatrix}.$$

Since  $X$  must be nilpotent of rank 2, it is clear that at least one of  $x_{14}$ ,  $x_{24}$ ,  $x_{34}$  is nonzero, and also at least one of  $x_{41}$ ,  $x_{42}$ ,  $x_{43}$  is nonzero. We may assume that  $x_{14} \neq 0$ . Then  $X^2 = 0$  implies that  $x_{42} = x_{43} = 0$ , and so  $x_{41} \neq 0$ . As  $x_{22}$  and  $x_{33}$  are eigenvalues of  $X$ , we must have  $x_{22} = x_{33} = 0$ . From  $X^2 = 0$  we deduce that  $x_{24} = x_{34} = 0$ . Thus only the four corner entries of  $X$  may be nonzero. As  $X$  is nilpotent of rank 2, we have a contradiction.  $\square$

**Proposition 7.2.** *The second pattern in Table 2 is not universal.*

*Proof.* Denote this pattern by  $I$  and let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & -i & 0 \\ 0 & -1 & 1 & 1-i \\ 1 & 2 & 0 & i-1 \end{bmatrix},$$

a nilpotent matrix of rank 3. Assume that  $AU = UX$  for some  $X = [x_{ij}] \in L_I$  and some  $U = [u_{ij}] \in U(4)$ . Thus  $X$  has the form

$$X = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ * & 0 & * & * \\ * & * & * & * \end{bmatrix}.$$

Let  $u_k$  denote the  $k$ -th column of  $U$ . By equating the entries in  $X = U^*AU$ , we obtain that  $x_{ij} = u_i^*Au_j$  for all  $i, j$ .

Since  $X$  is nilpotent, we must have  $x_{11} = 0$ . The first row of  $X$  is zero, and the other three rows must be linearly independent because  $X$  has rank 3. Since the first row of  $UX = AU$  is zero, we conclude that  $u_{12} = u_{13} = u_{14} = 0$ . As  $U$  is unitary, we also have  $u_{21} = u_{31} = u_{41} = 0$ . Without any loss of generality, we may assume that  $u_{11} = 1$ .

Since  $u_2^*Au_1 = x_{21} = 0$ , we have  $u_{42} = 0$ .

Assume that  $u_{32} = 0$ . Then  $u_{23} = u_{24} = 0$ , and we may assume that  $u_{22} = 1$ . Since  $u_3^*Au_2 = x_{32} = 0$ , we obtain that  $-\bar{u}_{33} + 2\bar{u}_{43} = 0$ . Since  $u_3 \perp u_4$ , we must have  $u_{44} = -2u_{34} \neq 0$ . Now we obtain a contradiction:  $0 = x_{24} = u_2^*Au_4 = -iu_{34}$ .

Thus we must have  $u_{32} \neq 0$ . Then  $u_2 = u_{32}(0, \xi, 1, 0)$  with  $\xi = u_{22}/u_{32}$ . If  $u_{24} = 0$ , then  $u_2 \perp u_4$  implies that  $u_{34} = 0$  and the condition

$x_{24} = 0$  again gives a contradiction. Consequently, we must have  $u_{24} \neq 0$  and so  $u_4 = u_{24}(0, 1, -\bar{\xi}, \eta)$  for some  $\eta \in \mathbf{C}$ . Then we have

$$\begin{aligned} Au_2 &= u_{32}(0, -i(1 + \xi), 1 - \xi, 2\xi), \\ Au_4 &= u_{24}(0, i(\bar{\xi} - 1), -1 - \bar{\xi} + (1 - i)\eta, 2 + (i - 1)\eta). \end{aligned}$$

As we must have  $Au_4 \perp u_2$ , we obtain that

$$(7.1) \quad i\xi(1 - \xi) - (1 + \xi) + (1 + i)\bar{\eta} = 0.$$

Since  $U^*Au_2$  is the second column of  $X$ ,  $Au_2$  must be a linear combination of  $u_2$  and  $u_4$ . Therefore

$$\begin{vmatrix} \xi & -i(1 + \xi) & 1 \\ 1 & 1 - \xi & -\bar{\xi} \\ 0 & 2\xi & \eta \end{vmatrix} = 0,$$

i.e.,

$$(7.2) \quad \eta(\xi(1 - \xi) + i(1 + \xi)) + 2\xi(1 + |\xi|^2) = 0.$$

From (7.1) and (7.2) we obtain that

$$\begin{aligned} \xi(1 - \xi) + i(1 + \xi) &= (i - 1)\bar{\eta}, \\ (1 - i)|\eta|^2 &= 2\xi(1 + |\xi|^2). \end{aligned}$$

Hence,  $\xi = \lambda(1 - i)$  for some real  $\lambda \geq 0$ .

It follows that

$$\begin{aligned} |\eta|^2 &= 2\lambda(1 + 2\lambda^2), \\ (i - 1)\bar{\eta} &= i + 2\lambda + 2i\lambda^2, \\ 2|\eta|^2 &= 4\lambda^4 + 8\lambda^2 + 1, \\ 0 &= (1 - 2\lambda + 2\lambda^2)^2, \end{aligned}$$

which is a contradiction.  $\square$

We now give the promised example of a pattern which is universal and singular.

**Proposition 7.3.** *The third pattern in Table 2 is universal.*

*Proof.* Let  $A$  be any linear operator on  $\mathbf{C}^4$  of trace 0. We have to construct an orthogonal basis  $\{a_1, a_2, a_3, a_4\}$  such that, with respect to this new basis, the matrix of  $A$  belongs to  $L_I$ .

For  $a_1$  we choose an eigenvector of  $A^*$ . The case when  $a_1$  is also an eigenvector of  $A$  is easy and we leave it to the reader. We extend  $\{a_1\}$

to an orthogonal basis  $\{a_1, b_1, b_2, b_3\}$  such that  $Aa_1 = \lambda a_1 + b_3$  for some  $\lambda \in \mathbf{C}$ . The matrix of  $A$  with respect to this new basis has the form

$$\left[ \begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & B & \\ 1 & & & \end{array} \right].$$

In order to complete the proof, it suffices to show that there exist nonzero vectors  $x, y \in a_1^\perp$  such that

$$(7.3) \quad Ax \perp y, \quad x \perp Ay, \quad x \perp y, \quad b_3 \in \text{span}\{x, y\}.$$

We shall now work with the  $A$ -invariant subspace  $a_1^\perp$ . Let  $b_{ij}$ ,  $i, j \in \{1, 2, 3\}$  be the entries of the submatrix  $B$ . We may assume that  $b_{31}\bar{b}_{23} \neq b_{32}\bar{b}_{13}$  because such matrices form a dense open subset of  $M(3)$ . We shall write vectors in  $a_1^\perp$  by using their coordinates with respect to the basis  $\{b_1, b_2, b_3\}$ . We shall seek the vectors  $x$  and  $y$  in the form

$$x = (a + ib, c + id, 1), \quad y = (a + ib, c + id, -a^2 - b^2 - c^2 - d^2),$$

where  $a, b, c, d \in \mathbf{R}$ . Observe that the last two conditions in (7.3) are automatically satisfied. The first two conditions give

$$(7.4) \quad \begin{aligned} & (b_{11}(a + ib) + b_{12}(c + id) + b_{13})(a - ib) + \\ & (b_{21}(a + ib) + b_{22}(c + id) + b_{23})(c - id) - \\ & (b_{31}(a + ib) + b_{32}(c + id) + b_{33})(a^2 + b^2 + c^2 + d^2) = 0 \end{aligned}$$

and

$$(7.5) \quad \begin{aligned} & (b_{11}(a + ib) + b_{12}(c + id) - b_{13}(a^2 + b^2 + c^2 + d^2))(a - ib) + \\ & (b_{21}(a + ib) + b_{22}(c + id) - b_{23}(a^2 + b^2 + c^2 + d^2))(c - id) + \\ & (b_{31}(a + ib) + b_{32}(c + id) - b_{33}(a^2 + b^2 + c^2 + d^2)) \cdot 1 = 0. \end{aligned}$$

By taking the difference and cancelling the factor  $1 + a^2 + b^2 + c^2 + d^2$ , we obtain the linear equation

$$(7.6) \quad b_{13}(a - ib) + b_{23}(c - id) - b_{31}(a + ib) - b_{32}(c + id) = 0.$$

Thus our problem is reduced to showing that the equations (7.4) and (7.6) have a real solution for the unknowns  $a, b, c, d$ . We now set  $b_{ij} = b'_{ij} + ib''_{ij}$  where  $b'_{ij}, b''_{ij} \in \mathbf{R}$  and denote by  $(S)$  the system of four equations obtained from (7.4) and (7.6) by equating to zero their real and imaginary parts. The first two of these four equations are not homogeneous. By homogenizing these two equations we obtain the system which we denote by  $(S')$ . Although we are interested in real solutions, we shall now consider all complex solutions of  $(S')$  in

the complex projective 4-space. By Bézout's theorem, there are 9 solutions in the generic case (counting multiplicities). We are going to show that exactly two of these solutions lie on the hyperplane at infinity. Consequently, the system  $(S)$  has exactly 7 solutions (counting multiplicities). Since the non-real solutions come in complex conjugate pairs, at least one of them has to be real. Clearly, this will complete the proof.

To find the solutions in the hyperplane at infinity (a complex projective 3-space), we have to solve yet another homogeneous system,  $(S'')$ , which is obtained from  $(S)$  by omitting the terms of degree less than 3 in the first two equations and retaining the last two (linear) equations. The two new cubic equations factorize as follows:

$$\begin{aligned} (a^2 + b^2 + c^2 + d^2)(b'_{31}a - b''_{31}b + b'_{32}c - b''_{32}d) &= 0, \\ (a^2 + b^2 + c^2 + d^2)(b''_{31}a + b'_{31}b + b''_{32}c + b'_{32}d) &= 0. \end{aligned}$$

If we assume that  $a^2 + b^2 + c^2 + d^2 \neq 0$ , then we obtain a system of four linear equations. The condition  $b_{31}\bar{b}_{23} \neq b_{32}\bar{b}_{13}$  is just saying that the determinant of this system of linear equations is not 0. Thus, this system has only the trivial solution. Hence the solutions of  $(S'')$  are just the solutions of the system of two linear equations of  $(S)$  and the equation  $a^2 + b^2 + c^2 + d^2 = 0$ . But a line intersects a quadric in exactly two points and we are done.  $\square$

For the remaining four exceptional classes the universality remains undecided.

**Problem 7.4.** Decide which of the last four patterns in Table 2 are universal.

## 8. COUNTING REDUCING FLAGS

Recall that the homogeneous space  $\mathcal{F}_n = \mathrm{U}(n)/T_n$  is known as the *flag variety*. It is a real smooth manifold of dimension  $n(n-1)$ . The points of this manifold can be interpreted in several ways. By the above definition, the points are cosets  $gT_n$ ,  $g \in \mathrm{U}(n)$ . They can be viewed also as complete flags

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbf{C}^n,$$

i.e., the increasing sequence of complex subspaces  $V_k$  with  $\dim V_k = k$ . We shall follow this practise and refer to the points of  $\mathcal{F}_n$  as *flags*. Yet another often used interpretation is to consider the points of  $\mathcal{F}_n$  as ordered  $n$ -tuples  $(W_1, W_2, \dots, W_n)$  of 1-dimensional complex subspaces

of  $\mathbf{C}^n$  which are orthogonal to each other under the standard inner product.

Let  $I \in \mathcal{P}'_n$  and  $A \in L(n)$ . Assume that the orbit  $\mathcal{O}_A$  meets  $L_I = L_I(n)$  and let  $\mathcal{X}_A = \mathcal{O}_A \cap L_I$ . The maximal torus  $T_n$  acts on  $\mathcal{X}_A$  (on the right) by conjugation, i.e.,  $(X, t) \rightarrow t^{-1}Xt$  where  $t \in T_n$  and  $X \in \mathcal{X}_A$ . Let us also introduce the set

$$\mathcal{U}_A = \{g \in \mathrm{U}(n) : g^{-1}Ag \in L_I\},$$

on which  $T_n$  acts by right multiplication. If  $g \in \mathcal{U}_A$  then  $g^{-1}Ag \in L_I$  and we say that the flag  $gT_n$  *reduces*  $A$  to  $L_I$ . Thus the set  $\mathcal{U}_A/T_n$  can be identified with the set of all reducing flags of the matrix  $A$ .

The map

$$(8.1) \quad \theta : \mathcal{U}_A \rightarrow \mathcal{X}_A, \quad \theta(g) = g^{-1}Ag,$$

is surjective and  $T_n$ -equivariant. Our main objective in this section is to prove that the induced map

$$(8.2) \quad \hat{\theta} : \mathcal{U}_A/T_n \rightarrow \mathcal{X}_A/T_n$$

is bijective, i.e., that we have a natural bijective correspondence between the reducing flags (for  $A$ ) and the  $T_n$ -orbits in  $\mathcal{X}_A$ .

We need three lemmas. The first one is valid for any connected compact Lie group  $G$ . For any  $g \in G$  let  $Z_g$  denote the identity component of the centralizer of  $g$  in  $G$ .

**Lemma 8.1.** *Let  $G$  be a connected compact Lie group and  $H$  a connected closed subgroup of maximal rank. For any  $g \in G$ ,  $g$  belongs to the center of  $Z_g$ . If  $g \in H$  then  $H$  contains the center of  $Z_g$ .*

*Proof.* Let  $T$  be a maximal torus of  $G$  such that  $g \in T$ . As  $T \subseteq Z_g$ , we have  $g \in Z_g$  and the first assertion follows. If  $g \in H$ , we may assume that  $T$  is chosen so that  $T \subseteq H$ . Then  $T$  is a maximal torus of  $Z_g$ , and so  $T$  must contain the center of  $Z_g$ . As  $T \subseteq H$ , we are done.  $\square$

The next two lemmas deal with the case  $G = \mathrm{U}(n)$ .

**Lemma 8.2.** *If  $H_1$  and  $H_2$  are connected closed subgroups of  $\mathrm{U}(n)$  of rank  $n$ , then  $H_1 \cap H_2$  is connected.*

*Proof.* In view of the above lemma, it suffices to show that the center of  $Z_g$  is connected for all  $g \in \mathrm{U}(n)$ . To prove this, we may assume that  $g$  is a diagonal matrix. It follows that

$$Z_g \cong \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_k), \quad n_1 + \cdots + n_k = n.$$

Hence the center of  $Z_g$  is a torus.  $\square$

**Lemma 8.3.** *The centralizer of any  $A \in M(n)$  in  $\mathrm{U}(n)$  is connected.*

*Proof.* Let  $A = A_1 + iA_2$  where  $A_1$  and  $A_2$  are hermitian matrices. Since  $A_k$  is unitarily diagonalizable, its centralizer  $H_k$  in  $U(n)$  is a closed connected subgroup of rank  $n$ . Hence, the centralizer  $H_1 \cap H_2$  of  $A$  in  $U(n)$  is connected by Lemma 8.2.  $\square$

We can now prove the desired result. Recall that  $A \in L(n)$  is  $I$ -generic if  $\mathcal{O}_A$  and  $L_I$  intersect transversally.

**Theorem 8.4.** *Let  $I \in \mathcal{P}'_n$  and let  $A \in L(n)$  be  $I$ -generic. Then the map  $\hat{\theta}$  defined by (8.1) and (8.2) is bijective. Consequently,  $N_{A,I}$  is the number of flags which reduce  $A$  to  $L_I$ .*

*Proof.* Since  $\theta$  is surjective, so is  $\hat{\theta}$ . In order to prove that  $\hat{\theta}$  is injective, it suffices to show that if  $g_1, g_2 \in \mathcal{U}_A$  are such that  $g_1^{-1}Ag_1 = g_2^{-1}Ag_2$ , we denote this matrix by  $B$ , then the element  $h = g_1^{-1}g_2$  belongs to  $T_n$ . If  $\mathfrak{u}(n) \subseteq M(n)$  is the space of skew-hermitian matrices, the transversality hypothesis implies that

$$\{X \in \mathfrak{u}(n) : [X, B] \in L_I\}$$

is the space  $\mathfrak{t}_n$  of the diagonal skew-hermitian matrices. Hence, the Lie algebra of the centralizer of  $B$  in  $U(n)$  is contained in  $\mathfrak{t}_n$ . By the above lemma this centralizer is connected, and so must be contained in  $T_n$ . As  $h$  commutes with  $B$ , we have  $h \in T_n$ .  $\square$

*Remark 8.5.* It follows from the theorem that the number  $N_{A,I}$  coincides with the number denoted by  $N(A, M_I(n, \mathbf{C}))$  in [1].

## 9. THE CYCLIC PATTERN IN THE CASE $n = 3$

In this section we consider only the case  $n = 3$  and the cyclic pattern

$$I = \{(1, 3), (2, 1), (3, 2)\} \in \mathcal{P}'_3.$$

By Theorem 3.8,  $I$  is universal. For  $A \in L(3)$  let  $\mathcal{O}_A = \{UAU^{-1} : U \in U(3)\}$  be its orbit. Recall that  $A$  (or  $\mathcal{O}_A$ ) is said to be  $I$ -generic if  $\mathcal{O}_A$  and  $L_I = L_I(3)$  intersect transversally. Since  $I$  is fixed, we shall drop the prefix “ $I$ ” and say just that  $A$  is generic. We denote by  $\Theta$  the set of nongeneric matrices in  $L(3)$ . Clearly  $\Theta$  is a closed  $U(n)$ -invariant subset.

We shall study here the set  $\Theta$ , and the intersections  $\mathcal{X}_A = \mathcal{O}_A \cap L_I$  for generic  $A \in L(3)$ . In particular, we are interested in the possible values of the number,  $N_A = N_{A,I}$ , of  $T_3$ -orbits contained in  $\mathcal{X}_A$ .

First we consider the intersection  $\Theta \cap L_I$ . This set contains all points  $A \in L_I$  such that  $\mathcal{O}_A$  and  $L_I$  are not transversal at  $A$ . An arbitrary

matrix  $A \in L_I$  can be written as

$$(9.1) \quad A = \begin{bmatrix} u & z & 0 \\ 0 & v & x \\ y & 0 & w \end{bmatrix}, \quad w = -u - v.$$

Define the homogeneous polynomial  $P_1 : L_I \rightarrow \mathbf{R}$  of degree 6 in the real and imaginary parts of the complex variables  $x, y, z, u, v$  by:

$$(9.2) \quad \begin{aligned} P_1(A) = & |(v-w)x^2|^2 + |(w-u)y^2|^2 + |(u-v)z^2|^2 \\ & + (|(v-w)x|^2 + |yz|^2) \cdot (|v|^2 + |w|^2 - 5|u|^2) \\ & + (|(w-u)y|^2 + |zx|^2) \cdot (|w|^2 + |u|^2 - 5|v|^2) \\ & + (|(u-v)z|^2 + |xy|^2) \cdot (|u|^2 + |v|^2 - 5|w|^2) \\ & + |(u-v)(v-w)(w-u)|^2. \end{aligned}$$

**Proposition 9.1.** *For  $A \in L_I$  as above,  $\mathcal{O}_A$  and  $L_I$  intersect non-transversally at  $A$  iff  $A$  lies on the real hypersurface  $\Gamma_1 \subset L_I$  defined by the equation  $P_1 = 0$ .*

*Proof.* The tangent space to  $\mathcal{O}_A$  at the point  $A$  consists of all matrices  $[A, X] = AX - XA$  with  $X^* = -X$  and  $\text{tr}(X) = 0$ . If  $X$  is diagonal, then  $[A, X] \in L_I$ . Let  $S$  be the subspace of skew-Hermitian matrices with all diagonal entries 0. We see immediately that  $\mathcal{O}_A$  and  $L_I$  are transversal at  $A$  iff  $L_I + [A, S] = L(3)$ . A routine computation shows that the latter condition is equivalent to the vanishing of a certain determinant, a homogeneous polynomial of degree 6. It is not hard to verify that this polynomial (unique up to a scalar factor) is  $P_1$ .  $\square$

**Corollary 9.2.** *We have*

$$(9.3) \quad \Gamma_1 \subseteq \Theta \cap L_I,$$

$$(9.4) \quad \Theta = \cup_{A \in \Gamma_1} \mathcal{O}_A.$$

*Proof.* The inclusion (9.3) is obvious. If  $B \in \Theta$  then there exists  $A \in \mathcal{X}_B$  such that  $\mathcal{O}_B$  and  $L_I$  intersect non-transversally at  $A$ . Hence  $A \in \Gamma_1$  and  $B \in \mathcal{O}_A$ . Thus (9.4) is valid.  $\square$

Our next objective is to construct the (unique) irreducible real hypersurface  $\Gamma$  in  $L(3)$  containing the set  $\Theta$ . It is given by an equation  $P = 0$ , where  $P : L(3) \rightarrow \mathbf{R}$  is a homogeneous polynomial of degree 24 in 16 variables, the real and imaginary parts of the entries of  $X \in L(3)$  except the last entry. The polynomial  $P$  is invariant under the action of the direct product  $U(1) \times SU(3)$ , where  $U(1)$  acts by multiplication with scalars of unit modulus and  $SU(3)$  acts by conjugation. It can be

expressed as a polynomial in the invariants  $i_1, i_2, \dots, i_{16}$  listed in Appendix A. As this expression has 203 terms, we shall give it separately in Appendix B.

We warn the reader that  $P$  is rather large. Denote by  $P_I$  its restriction to the subspace  $L_I$ . We run out of memory if we try to evaluate  $P_I$  at an arbitrary matrix in  $L_I$  and expand it as a polynomial in the 10 real variables (the real and imaginary parts of the complex variables  $x, y, z, u, v$ ). However, when we set the imaginary parts of  $y$  and  $z$  to 0, then we can expand  $P_I$  and obtain 130571 terms. If we additionally set the imaginary part of  $x$  to 0, then the number of terms goes down to 50583.

Recall that the hypersurface  $\Gamma \subset L(3)$  is defined by the equation  $P = 0$ .

**Proposition 9.3.** *The restriction  $P_I$  admits the factorization:*

$$(9.5) \quad P_I = P_1^2 P_2,$$

where  $P_1$  is defined by (9.2) and  $P_2$  is a homogeneous polynomial of degree 12 with integer coefficients. Thus  $\Gamma \cap L_I = \Gamma_1 \cup \Gamma_2$ , where  $\Gamma_2$  is the hypersurface defined by  $P_2 = 0$ . We also have  $\Theta \subseteq \Gamma$ .

*Proof.* The first assertion can be verified by using a computer and suitable software, e.g., MAPLE [6]. The assertion that  $\Gamma \cap L_I = \Gamma_1 \cup \Gamma_2$  is now obvious. Finally, the assertion  $\Theta \subseteq \Gamma$  follows from Corollary 9.2 and the fact that  $\mathcal{O}_A \subseteq \Gamma$  for  $A \in \Gamma$ .  $\square$

We prove next that  $P_1$ ,  $P_2$  and  $P$  are irreducible. In fact we have a stronger result.

**Proposition 9.4.** *The real polynomials  $P_1$ ,  $P_2$  and  $P$  defined above are absolutely irreducible (i.e., irreducible over  $\mathbf{C}$ ).*

*Proof.* The absolute irreducibility of  $P_1$  can be proved by using the “absfact.lib” library in SINGULAR [2]. To make this computation easier, it suffices to check that after setting  $y = 2$ ,  $z = 1$ , and setting the imaginary parts of  $x$  and  $v$  to 0 and 1, respectively, the resulting polynomial (having 69 terms) still has degree 6 and is absolutely irreducible.

The same method works for  $P_2$ . In this case we set  $u = 1$ ,  $y = 2$ ,  $z = 1$ , and we also set the imaginary parts of  $x$  and  $v$  to 0 and 1, respectively. We obtain an absolutely irreducible polynomial of degree 12 (having 47 terms).

It is much harder to prove that  $P$  is also absolutely irreducible. (We were not successful in using the same method as above.) Assume that  $P$  has a nontrivial factorization  $P = QR$  over  $\mathbf{C}$ . We can assume that  $Q$  is irreducible. Since  $U(3)$  is connected and  $P$  is  $U(3)$ -invariant, both



$Q$  and  $R$  must be invariant. By restricting these polynomials to  $L_I$  and by using (9.5), we obtain that  $P_I = Q_I R_I = P_1^2 P_2$ . It follows that  $Q_I$  is equal to  $P_1$  or  $P_2$  (up to a scalar factor). We are going to show that this leads to a contradiction.

One can easily verify that, for

$$A = \begin{bmatrix} 3+3i & 5 & 0 \\ 0 & 3-3i & 5 \\ 5 & 0 & -6 \end{bmatrix} \in L_I,$$

we have  $P_1(A) = 0$  and  $P_2(A) = -22675690800$ , and so  $A \in \Gamma_1 \setminus \Gamma_2$ .

On the other hand, the matrix

$$B = \begin{bmatrix} -1 & \frac{1}{2}\sqrt{222+6\sqrt{69}} & 0 \\ 0 & \frac{1}{2}(1-\sqrt{69}) & 0 \\ \frac{1}{2}\sqrt{222-6\sqrt{69}} & 0 & \frac{1}{2}(1+\sqrt{69}) \end{bmatrix} \in L_I$$

satisfies  $P_1(B) = 89424$  and  $P_2(B) = 0$ , and so  $B \in \Gamma_2 \setminus \Gamma_1$ .

We obtain a contradiction by showing that  $B \in \mathcal{O}_A$ . An explicit unitary matrix  $X$  satisfying  $AX = XB$  is given in Appendix C.  $\square$

We note that, in the above proof,  $A$  is a regular point of  $\Gamma_1$ , while  $B$  is singular on  $\Gamma_2$ .

The subspace  $L_I$  is  $Z$ -invariant, for the cyclic matrix

$$Z = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Indeed, if  $A \in L_I$  is given by (9.1) then

$$ZAZ^{-1} = \begin{bmatrix} w & y & 0 \\ 0 & u & z \\ x & 0 & v \end{bmatrix} \in L_I.$$

Assume now that  $A \in L_I$  and that  $P(A) \neq 0$ . We claim that the two points  $A, ZAZ^{-1} \in \mathcal{X}_A$  are not  $T_3$ -conjugate. Otherwise  $ZAZ^{-1} = DAD^{-1}$  for some  $D \in T_3$ . This implies that  $u = v = w$ , and  $u + v + w = 0$  forces that  $u = v = w = 0$ . Consequently,  $P_1(A) = 0$  which contradicts our assumption. We conclude that the three points of the  $Z$ -orbit  $\{A, ZAZ^{-1}, Z^{-1}AZ\}$  belong to three different  $T_3$ -orbits in  $\mathcal{X}_A$ . Since  $P_1$  is  $Z$ -invariant, we have

$$P_1(A) = P_1(ZAZ^{-1}) = P_1(Z^{-1}AZ).$$

It follows that  $N_A$  is divisible by 3. As  $N_A = N(A, M_I(3))$  by the remark 8.5 and as we know that in this case  $N(A, M_I(3))$  is even, we

conclude that  $N_A$  is divisible by 6. Numerical computations indicate that  $N_A$  is either 6 or 18 and that  $\mathcal{X}_A/T_3$  can be split into 6-tuples such that  $P_1$  is constant on the union of the  $T_3$ -orbits belonging to the same 6-tuple. We have no explanation for this phenomenon.

Let us look at an example. The matrices

$$A = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -i & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1+i \end{bmatrix}$$

belong to  $L_I$ . As  $P_1(A) = P_1(B) = 45$  and  $P_2(A) = P_2(B) = 0$ , we have  $A, B \in \Gamma_2 \setminus \Gamma_1$ . One can easily verify that  $A$  and  $B$  are unitarily similar and are regular points of  $\Gamma_2$ . Hence  $\mathcal{X}_A$  contains the six  $T_3$ -orbits with representatives  $Z^k A Z^{-k}, Z^k B Z^{-k}$ ,  $k = 0, 1, 2$ . We propose  $A$  and  $B$  as examples of generic points which belong to  $\Gamma$ , which would imply that  $\Theta \neq \Gamma$ .

## 10. APPENDIX A: GENERATORS OF $U(1) \times SU(3)$ -INVARIANTS

Consider the representation of the direct product  $U(1) \times SU(3)$  on  $L(3)$ , where  $U(1)$  acts by multiplication with scalars of unit modulus and  $SU(3)$  acts by conjugation. The algebra of real polynomial invariants for this action is a subalgebra of the corresponding algebra for the conjugation action of  $U(3)$  on  $L(3)$ . A minimal set of homogeneous generators of the first algebra consists of 16 invariants  $i_1, i_2, \dots, i_{16}$  listed below. This fact is neither proved nor used in this paper. However, these generators are used in Sections 2 and 9 to construct some other

invariants that we need. The 16 generators are

$$\begin{aligned}
i_1 &= \operatorname{tr}(XY), i_2 = \operatorname{tr}(X^2Y^2), i_3 = \frac{1}{4}|\operatorname{tr}(X^2)|^2, \\
i_4 &= \operatorname{tr}(XYX^2Y^2), i_5 = \frac{1}{9}|\operatorname{tr}(X^3)|^2, i_6 = |\operatorname{tr}(X^2Y)|^2, \\
i_7 &= \frac{1}{6}\Re\left(\operatorname{tr}(Y^2)(3\operatorname{tr}^2(X^2Y) + \operatorname{tr}(X^3)\operatorname{tr}(XY^2))\right), \\
i_8 &= \frac{1}{6}\Re\left(\operatorname{tr}(Y^2)(3\operatorname{tr}^2(X^2Y) - \operatorname{tr}(X^3)\operatorname{tr}(XY^2))\right), \\
i_9 &= \frac{1}{2}\Im\left(\operatorname{tr}(Y^2)\operatorname{tr}^2(X^2Y)\right), \\
i_{10} &= \frac{1}{6}\Im\left(\operatorname{tr}(X^2)\operatorname{tr}^2(X^2Y)\operatorname{tr}(Y^3)\right), \\
i_{11} &= \frac{1}{12}\Re(\operatorname{tr}^2(X^2)\operatorname{tr}(Y^3)\operatorname{tr}(XY^2)), \\
i_{12} &= \frac{1}{12}\Im(\operatorname{tr}^2(X^2)\operatorname{tr}(Y^3)\operatorname{tr}(XY^2)), \\
i_{13} &= \frac{1}{72}\Re(\operatorname{tr}^2(X^3)\operatorname{tr}^3(Y^2)), \\
i_{14} &= \frac{1}{72}\Im(\operatorname{tr}^2(X^3)\operatorname{tr}^3(Y^2)), \\
i_{15} &= \frac{1}{3}\Im\left(\operatorname{tr}^3(X^2Y)\operatorname{tr}(Y^3)\right), \\
i_{16} &= \frac{1}{144}\Re(\operatorname{tr}^4(X^2)\operatorname{tr}^2(Y^3)\operatorname{tr}^2(XY^2)),
\end{aligned}$$

where  $X \in L(3)$  is an arbitrary matrix,  $Y = X^*$  is its adjoint, and  $\operatorname{tr}$  is the matrix trace function. The symbols  $\Re$  and  $\Im$  stand for “real part” and “imaginary part”, respectively.

## 11. APPENDIX B: THE POLYNOMIAL $P$

Here we construct the polynomial  $P$  used in Section 9. It will be given as a polynomial in the invariants  $i_k$  listed in Appendix A. For convenience, we collect  $P$  with respect to the invariants  $i_3$  and  $i_6$

$$(11.1) \quad P = \sum_{k,l} p_{kl} i_3^k i_6^l.$$

The nonzero  $p_{kl}$  are as follows:

$$\begin{aligned}
p_{00} &= -6 (126 i_2^2 i_1^4 - 26 i_2 i_1^6 + 336 i_7 i_2 i_1^2 - 270 i_2^3 i_1^2 + 1536 i_7^2 \\
&\quad + 216 i_2^4 - 11 i_7 i_1^4 + 2 i_1^8 - 1152 i_7 i_2^2) (i_8 + i_7) \\
p_{01} &= 18432 i_{11} i_7 + 186 i_8 i_1^5 + 2016 i_{11} i_1^2 i_2 + 2160 i_2^2 i_1 i_7 - 66 i_{11} i_1^4 \\
&\quad + 2160 i_8 i_1 i_2^2 - 1278 i_8 i_1^3 i_2 + 186 i_1^5 i_7 - 6912 i_{11} i_2^2 + 3456 i_1 i_7^2 \\
&\quad + 3456 i_8 i_1 i_7 - 1278 i_1^3 i_2 i_7 \\
p_{02} &= 3888 i_8 i_2 + 297 i_1^2 i_2^2 - 4608 i_{13} - 324 i_2^3 - 90 i_1^4 i_2 + 3888 i_2 i_7 \\
&\quad + 9 i_1^6 - 1242 i_1^2 i_7 - 1242 i_8 i_1^2 - 3456 i_{11} i_1 \\
p_{03} &= 18 i_1 (-7 i_1^2 + 27 i_2) \\
p_{04} &= 729 \\
p_{10} &= 6912 i_7 i_4 i_1 i_2 + 2016 i_2 i_1^2 i_{13} - 33 i_4 i_2 i_1^5 + 1008 i_4 i_2^2 i_1^3 \\
&\quad - 3456 i_4 i_1 i_2^3 - 1008 i_4^2 i_2 i_1^2 + 31104 i_5^2 i_2^2 + 33 i_5 i_1^7 \\
&\quad + 297 i_5^2 i_1^4 + 9792 i_2^3 i_8 - 2304 i_4^2 i_8 - 251 i_1^6 i_8 - 20736 i_5^2 i_8 \\
&\quad - 3024 i_5 i_4 i_2 i_1^2 + 25920 i_7 i_2^3 - 55296 i_7^2 i_2 - 6912 i_7 i_4^2 \\
&\quad + 17760 i_7^2 i_1^2 + 3456 i_4^2 i_2^2 + 9504 i_5 i_2^2 i_1^3 - 6912 i_7 i_5 i_1^3 \\
&\quad - 66 i_1^4 i_{13} + 4608 i_{16} - 24012 i_7 i_2^2 i_1^2 - 317 i_7 i_1^6 - 62208 i_7 i_5^2 \\
&\quad + 2304 i_4 i_2 i_1 i_8 - 1206 i_5 i_2 i_1^5 + 3012 i_2 i_1^4 i_8 + 5226 i_7 i_2 i_1^4 \\
&\quad - 20736 i_7 i_5 i_4 + 8544 i_7 i_1^2 i_8 - 6912 i_2^2 i_{13} + 9216 i_7 i_{13} + 66 i_1^5 i_{11} \\
&\quad - 9072 i_5^2 i_2 i_1^2 - 2304 i_5 i_1^3 i_8 - 27648 i_7 i_2 i_8 - 6912 i_5 i_4 i_8 \\
&\quad - 11052 i_2^2 i_1^2 i_8 + 13824 i_5 i_2 i_1 i_8 + 10368 i_5 i_4 i_2^2 + 1632 i_2^2 i_1^6 \\
&\quad - 5151 i_2^3 i_1^4 + 7200 i_2^4 i_1^2 - 256 i_2 i_1^8 + 16 i_1^{10} - 1728 i_2^5 \\
&\quad - 2016 i_2 i_1^3 i_{11} + 41472 i_7 i_5 i_1 i_2 - 18432 i_7 i_1 i_{11} + 6912 i_2^2 i_1 i_{11} \\
&\quad - 20736 i_5 i_1 i_2^3 + 99 i_5 i_4 i_1^4 + 33 i_4^2 i_1^4 \\
p_{11} &= -4320 i_5 i_2 i_1^2 + 6 i_1^3 i_8 + 6912 i_7 i_4 + 1728 i_4^2 i_1 - 3450 i_7 i_1^3 \\
&\quad - 5088 i_1^2 i_{11} + 14688 i_1 i_2^3 + 2721 i_2 i_1^5 + 2304 i_4 i_8 - 4320 i_2 i_1 i_8 \\
&\quad + 15552 i_5^2 i_1 + 1152 i_1 i_{13} + 11520 i_5 i_8 - 3456 i_4 i_2^2 + 1440 i_7 i_1 i_2 \\
&\quad - 20736 i_5 i_2^2 - 272 i_1^7 + 1530 i_5 i_1^4 - 33 i_4 i_1^4 + 32256 i_2 i_{11} \\
&\quad + 5184 i_5 i_4 i_1 - 9792 i_2^2 i_1^3 - 720 i_4 i_2 i_1^2 + 39168 i_7 i_5 \\
p_{12} &= -1728 i_4 i_1 + 2130 i_1^4 - 10170 i_2 i_1^2 + 15228 i_2^2 + 1296 i_8 \\
&\quad + 1296 i_7 - 10368 i_5 i_1 \\
p_{13} &= -3726 i_1
\end{aligned}$$

$$\begin{aligned}
p_{20} &= -4272 i_4 i_2 i_1^3 + 16128 i_4 i_1 i_2^2 - 16128 i_4^2 i_2 + 12816 i_5 i_4 i_1^2 \\
&\quad - 48384 i_5 i_4 i_2 + 4272 i_4^2 i_1^2 - 27360 i_2^2 i_8 + 23808 i_7 i_8 \\
&\quad + 80844 i_7 i_2 i_1^2 + 69888 i_7^2 - 47808 i_5 i_2 i_1^3 + 117504 i_5 i_1 i_2^2 \\
&\quad - 39168 i_7 i_5 i_1 + 19500 i_2 i_1^2 i_8 - 8544 i_1^2 i_{13} + 32256 i_2 i_{13} \\
&\quad + 8544 i_1^3 i_{11} + 384 i_1^8 + 20160 i_2^4 - 128736 i_7 i_2^2 - 11607 i_7 i_1^4 \\
&\quad + 4470 i_5 i_1^5 - 145152 i_5^2 i_2 + 38448 i_5^2 i_1^2 - 2799 i_1^4 i_8 \\
&\quad - 11520 i_5 i_1 i_8 - 50016 i_2^3 i_1^2 - 5279 i_2 i_1^6 + 26283 i_2^2 i_1^4 \\
&\quad - 32256 i_2 i_1 i_{11} \\
p_{21} &= -22224 i_7 i_1 - 4272 i_4 i_1^2 - 37632 i_{11} + 16128 i_4 i_2 + 96768 i_5 i_2 \\
&\quad - 2112 i_1^5 + 18192 i_2 i_1^3 - 15264 i_5 i_1^2 - 8400 i_1 i_8 - 40608 i_1 i_2^2 \\
p_{22} &= -10476 i_2 + 10401 i_1^2 \\
p_{30} &= 44448 i_5 i_1^3 + 31296 i_8 i_2 + 4415 i_1^6 - 76308 i_1^2 i_7 - 18816 i_4 i_1 i_2 \\
&\quad + 169344 i_5^2 + 56448 i_5 i_4 + 128496 i_1^2 i_2^2 + 37632 i_{11} i_1 \\
&\quad - 209664 i_5 i_1 i_2 + 18816 i_4^2 - 9108 i_8 i_1^2 + 236352 i_2 i_7 \\
&\quad - 37632 i_{13} - 90624 i_2^3 - 43032 i_1^4 i_2 \\
p_{31} &= -117504 i_5 + 480 i_2 i_1 + 3312 i_1^3 - 18816 i_4 \\
p_{32} &= -5196 \\
p_{40} &= -146816 i_2 i_1^2 - 10384 i_8 + 112896 i_5 i_1 + 23948 i_1^4 \\
&\quad + 195840 i_2^2 - 142480 i_7 \\
p_{41} &= 33632 i_1 \\
p_{50} &= -202048 i_2 + 65232 i_1^2 \\
p_{60} &= 78400
\end{aligned}$$

12. APPENDIX C: THE UNITARY MATRIX  $X$ 

The entries of the unitary matrix  $X = [x_{ij}]$ , used in the proof of Proposition 9.4, are given by

$$\begin{aligned}
x_{11} &= \frac{2(\sqrt{69} - 7) - i(3 + \sqrt{69})}{6\sqrt{37 - \sqrt{69}}}, \\
x_{12} &= \frac{1}{12\sqrt{13}}(1 + 3i)(i\sqrt{6} - \sqrt{46}), \\
x_{13} &= \frac{1}{12}(\sqrt{46} + \sqrt{6}), \\
x_{21} &= \frac{(3 - i)\sqrt{69} + 34 + 27i}{15\sqrt{37 - \sqrt{69}}}, \\
x_{22} &= \frac{4}{15\sqrt{13}}(\sqrt{46} - \sqrt{6}) + \frac{i}{60\sqrt{13}}(23\sqrt{6} - 3\sqrt{46}), \\
x_{23} &= \frac{1}{60}(3 - i)(3\sqrt{6} - i\sqrt{46}), \\
x_{31} &= \frac{2(2 - \sqrt{69}) - i(3 + \sqrt{69})}{6\sqrt{37 - \sqrt{69}}}, \\
x_{32} &= \frac{4}{15\sqrt{13}}(\sqrt{6} + \sqrt{46}) + \frac{i}{60\sqrt{13}}(23\sqrt{6} + 3\sqrt{46}), \\
x_{33} &= \frac{1}{12}(\sqrt{46} - \sqrt{6}).
\end{aligned}$$

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